You can start

docker pull registry.gitlab.inria.fr/soliman/inf555/td3
now
Local Search & Constraint Satisfaction Problems

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Thanks to P. Flener, L. Michel and P. Van Hentenryck for inspiration
What is “Local Search”

- iterative optimization method
- looking for an assignment of variables to values of their domains that minimizes some cost
- local move from solution to neighboring solution
- try to (always or not) decrease the cost of the selected assignment
How does this relate to Constraint Solving?

Compared to what we have seen up to now:

- only optimization

- requires defining:
  - neighborhood and
  - selection criterion (*single* state)
  - stopping criterion

⇒ **incomplete** (i.e., not optimal) but **low cost** (time and memory)
Constraint-Based Local Search

A pure CSP can be transformed easily into a LS problem:

Use constraint violations as cost function. Some constraints can be given an infinite cost, these are hard constraints that all states have to satisfy. Other are soft constraints that will guide the search. Allows us to solve over-constrained problems. Neighborhood is defined as changing a single variable assignment. In that framework, redundant constraints play two roles: propagation and search.
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Other are soft constraints that will guide the search.
Allows us to solve *over-constrained* problems.

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Allows us to solve over-constrained problems

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In that framework redundant constraints play two roles: propagation and search
Violations

Violations defined on basic constraints

Can be composed:

- $V(c_1 \land c_2) = V(c_1) + V(c_2)$
- $V(c_1 \lor c_2) = \min(V(c_1), V(c_2))$

- $V(\neg c) = 1 - \min(1, V(c))$
Violations

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Can be \textit{composed}:

- \( V(c_1 \land c_2) = V(c_1) + V(c_2) \)
- \( V(c_1 \lor c_2) = \min(V(c_1), V(c_2)) \)

- \( V(\neg c) = 1 - \min(1, V(c)) \) \( \Rightarrow \) not compatible with the above!

For global constraints, one can \textit{decompose} or not
N-Queens as a Local Search problem

```plaintext
constraint alldifferent(q);
constraint alldifferent([q[i] + i | i in 1..n]);
constraint alldifferent([q[i] - i | i in 1..n]);
```

Count violations as the total number of identical pairs in an alldifferent constraint

Very dependent on the model! (dual, symmetries, etc.)
But the state space does depend too!
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Very dependent on the model! (dual, symmetries, etc.)
But the state space does depend too!

Could use the max number of equalities instead of their sum
try to guide the search as much as possible
alldifferent lists:

→

Neighborhood:
Move one queen in its column i.e., change the valuation of a single variable.
All different lists:

\[
\rightarrow [4, 3, 4, 3] \quad \searrow
\]
alldifferent lists:
→[4,3,4,3] ↘[4,4,6,6] ↗

Violations: 2+ 2+ 1=5

Neighborhood:
Move one queen in its column i.e., change the valuation of a single variable
alldifferent lists:
→[4, 3, 4, 3] ↘[4, 4, 6, 6] ↗[4, 2, 2, 0]

Violations:
alldifferent lists:
→ [4, 3, 4, 3] ↘ [4, 4, 6, 6] ↗ [4, 2, 2, 0]

Violations:
2+
\textbf{alldifferent lists:}

\[
\rightarrow [4, 3, 4, 3] \quad \downarrow [4, 4, 6, 6] \quad \uparrow [4, 2, 2, 0]
\]

Violations:

\[2 + 2+\]
alldifferent lists:
→[4,3,4,3]  \[4,4,6,6\]  ↗[4,2,2,0]

Violations:
2+2+1 = 5

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Violations:

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Neighborhood:

Move \textbf{one queen} in its column

i.e., change the valuation of a single variable
Example: Greedy Local Search (aka. Hill-climbing)

Pure exploitation (intensification)

Analog for the discrete case of gradient descent

Select the most improving neighbor (random if multiple bests)

Stop when no improvement found
Example: Min. Conflict Search (MCS) / Heuristics (MCH)

Original heuristics for CBLS on SAT problems (and still part of GSAT, WalkSAT, etc.)

Implemented by default in the COMET system

Basis of most other heuristics

Select the neighbor (i.e., variable assignment) that minimizes the number of violated constraints
Example: Min. Conflict Search (MCS) / Heuristics (MCH)

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Basis of most other heuristics

Select the neighbor (i.e., variable assignment) that minimizes the **number** of violated constraints

IOW, Hill-Climbing with violations saturated at 1
Issues

Local extrema

Plateaus

Diagonal ridges (i.e., moves of same cost leading to different extrema)

Big neighborhood (might require two steps: variable and then value selection)
Issues

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Diagonal ridges (i.e., moves of same cost leading to different extrema)

Big neighborhood (might require two steps: variable and then value selection)
Violation cost =
Violation cost = 1

Line-wise costs
Violation cost = 1

Line-wise costs

Column-wise costs
Line-wise costs

Real local optimum

Necessary to get through a worse solution to get to a global optimum
Example: Random walk

Pure **exploration** (diversification)

Select a *random* neighbor

Remember the best solution found

Stop after a given number of iterations
Being smarter

⇒ combine a way to escape local extrema with some Hill-climbing

Some examples:
  - diagonal moves (see Practical work session)
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- Simulated annealing (idem)
- Tabu search
- random restarts (underrated!)
Being smarter

⇒ combine a way to escape local extrema with some Hill-climbing

Some examples:
- diagonal moves (see \textit{Practical work session})
- Simulated annealing (\textit{idem})
- Tabu search
- random \texttt{restarts} (underrated!)

What if \textit{restarts} were actually done simultaneously? population-based approaches
Simulated annealing

Inspiration from the physics’ world (Metropolis-Hastings algorithm for a sampling states of a thermodynamic system, 1953)

Allow some exploration while the temperature of the system is high

Decrease temperature with time (i.e., iterations)

Focus on exploitation when the system cools down
Practically

At each step:

- select a **random neighbor** (no guidance at all...)
- compare its cost with the current cost
- accept it or not depending on the temperature but **always** accept improving moves
- stop if the move was rejected and the temperature too low (return the best solution found)

Parameters: initial/stopping temperature, cooling regime, acceptance condition
Parameters

Initial temperature:
Parameters

Initial temperature: allow any move (random-walk)

Ending temperature:
Parameters

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Ending temperature: would reject most non-improving moves (hill-climbing)

Cooling regime:
Parameters

Initial temperature: allow any move (random-walk)

Ending temperature: would reject most non-improving moves (hill-climbing)

Cooling regime: observed to have almost no impact usually $T_{t+1} = (1 - c)T_t$ with a small cooling-rate, e.g., $c = 0.01$

Acceptance condition:
Parameters

Initial temperature: allow any move (random-walk)

Ending temperature: would reject most non-improving moves (hill-climbing)

Cooling regime: observed to have almost no impact
usually $T_{t+1} = (1 - c)T_t$ with a small cooling-rate, e.g., $c = 0.01$

Acceptance condition: mostly coming from physics consensus: $\exp(\Delta/T) > r$, $\Delta$ is current cost minus new cost, $r$ is a random variable in $[0,1)$
Results

Very often used as a basic LS algorithm

Decent results on the Traveling Salesman Problem not for finding an optimal solution but a *good* solution

On the N-Queens problem...
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Decent results on the Traveling Salesman Problem
not for finding an optimal solution but a *good* solution

On the N-Queens problem...
we’ll see during the *practical work session*

Not always easy to fine-tune temperature
Tabu search

Created by F. Glover in 1986

At its core: Hill-Climbing with some kind of memory forbidding moves

At each step select the best neighbor that is not tabu

Stop after a given number of iterations

Tabu moves force diversification
Tabu search

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Tabu moves force diversification in a more guided way than random-walks
Types of memory

The *Tabu-list* is usually a list of recently visited states, to avoid cycles (classical issue with random diagonal moves)

Its length is a **sensitive parameter**

Sometimes not a full state but a *feature* is stored

in that case it might be necessary to overcome *tabu* when a better candidate is found

Intermediate-term memory (intensification) and Long-term memory (diversification) rules can be added
Results

Same as SA but better/worse

More *guided*, but more parameters

Good results on TSP

Even trickier to fine-tune (especially complex Tabu structures/rules)
Population-based approaches

Many versions: Genetic algorithms, Particle Swarm Optimization, Ant-Colony Optimization, etc.

Main idea: instead of restarts, use the information of parallel runs while they are running

Balance between exploration/exploitation, diversification/intensification remains hard
Conclusion

Incomplete but **efficient** method, even with *simple algorithms* (e.g., Hill-climbing with restarts and diagonal moves)

Used for **hard** problems for which a complete search is not tractable (e.g., Ant Colony on graph problems by C. Solnon)

Or for problems that are **over-constrained** (and can be expressed as optimization) (e.g., MaxSAT)

Can be made generic for CSPs once violations are defined

Remains often tricky to parametrize
Does **never prove optimality**
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