Constraint-based Modeling and Algorithms for Decision-making — INF555

Sylvain Soliman

informatics mathematics

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François Fages, Sylvain Soliman
Project-team Lifeware — Inria Saclay
http://lifeware.inria.fr/

Part I

Decision problems, optimization, complexity and modelling

Decision Problems

- Finite input
  - Words
  - Rational numbers
  - Images
  - Sounds
  - Programs
  - ...

- **yes/no** output
  - providing an arbitrary solution is optional
  - providing *all solutions* is another class: **enumeration problem**

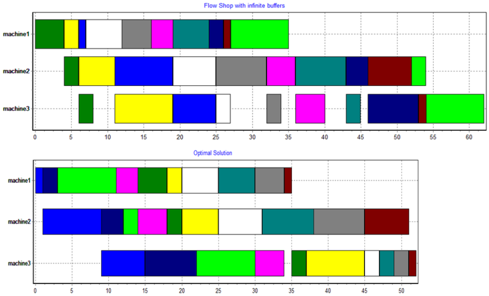
Examples

We will see during the class:

- Can we place N queens on a chessboard with no attack?
- Do we have enough rooms for the lectures? **assignment problem**
- Can we land in the next 20 mn a given set of flights arriving in Orly? **scheduling**
- Can we find a sequence of actions to achieve a given goal? **planning**
- ...

Other classical examples include **routing** (traveling salesman), **personnel staffing**, etc.

Optimization Problems



- Input: finite data same as before

- output: **optimal cost**
  - single number, or
  - vector of numbers for *multi-objective* optimization
  - providing an arbitrary optimal solution is optional
  - providing *all solutions* is another class: **enumeration problem**

- How many rooms are necessary to give the lectures?
- What is the time required to land a given set of flights arriving in Orly?
- How many flights from a given set of flights can land in the next 20min?
- ...

An optimization problem admits one **associated decision problem**: Is there a solution of given cost k?

Several optimization problems can be associated to a decision problem (choice of the inputs treated as objective function)

Mono-objective vs. Multi-objective optimization

The cost is always finite data

An *unique* rational number for mono-objective optimization.

- single optimization criterion
- possibly a robustness criterion w.r.t. perturbations
- the aggregation of multiple criteria representing a **trade-off**

Mono-objective vs. Multi-objective optimization

Some finite vector of rational numbers for multi-objective optimization

$$(f, g) < (f', g') \triangleq (f < f' \land g \leq g') \lor (f \leq f' \land g < g')$$

The optimal cost vectors $\max(f, g)$ not unique: **Pareto frontier**

Computational Time-complexity Classes

An algorithm belongs to time-complexity class C if it requires **at most** $C(n)$ elementary operations on a random access machine for an **input of size** n

A problem belongs to class C if there exists an algorithm in C to solve it

$$g(n) \in O(f(n)) \triangleq \exists k, n_0 \quad \forall n > n_0 \quad g(n) \leq k \cdot f(n)$$

Examples

$O(1)$	constant time, independent of the input
$O(n)$	linear time
$O(n \log n)$	best **possible** complexity for sorting
$O(n^2)$	quadratic time
P / PTIME	**polynomial time**, i.e. $O(n^k)$ for some k
$O(2^n)$	
EXPTIME	exponential time, i.e. $O(k^n)$ for some k
$O(2^{2^n})$	
Non-elem.	not bounded by a finite tower of exponentials $O(2^{2^{2^{\cdots}}})$

Non-deterministic Time-complexity

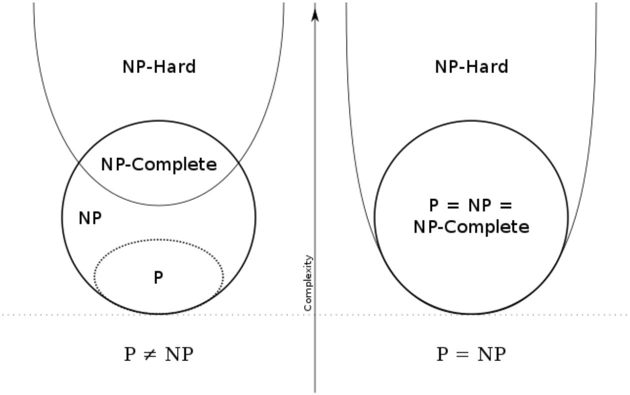
NP: class of languages recognized in polynomial time by a **non-deterministic** Turing machine
≡ decision problems with proofs **verifiable in PTIME**

E.g., disjunctive scheduling, timetabling, planning, …

NP-complete: class of problems **in NP** that can encode in polynomial time **any other NP problem**, i.e., the *hardest* NP problems

E.g., boolean satisfiability, graph coloring, disjunctive scheduling, …

NP-hard: class of problems *harder than NP*, i.e., they can encode in polynomial time any NP problem



E.g., *optimization problems* with NP-complete associated decision problem

Space-complexity Classes

An algorithm belongs to a space complexity class C if it requires **at most** C memory locations
PSPACE: polynomial space, i.e., $O(n^k)$ for some k

Why do we have NP$\subset$PSPACE?

Space-complexity Classes

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Why do we have NP⊂PSPACE?

Iterative deepening!
Bounded backtracking with increasing depth d
At depth d, d binary choices to remember, $O(d)$ space
Computation of result at $O(n^k)$ depth: $O(n^k)$ space

Time Complexity ≠ Size of the Search Space

Size of the solution domain:
bad upper bound for time-complexity

Sorting n integers	$!n$	$O(n \log n)$
Placing n queens	n^n or $!n$	

Time Complexity $\neq$ Size of the Search Space

Size of the solution domain:
bad upper bound for time-complexity

Sorting n integers	$!n$	$O(n \log n)$
Placing n queens	n^n or $!n$	$O(1)$ analytic solution
Fermat Theorem	∞	$O(1)$
$\forall n \exists ? abc\ a^n + b^n = c^n$		

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Fermat Theorem	∞	$O(1)$
$\forall\, n \exists ? abc\ a^n + b^n = c^n$		$n >? 2$ (A. Wiles 1994)

Separation results (like $\mathbf{P} \neq \mathbf{NP}$) are hard:
need to quantify on all mathematical properties

Theoretical Complexity only Bounds the Practical Complexity

Worst-case complexity $\neq$ **average** time-complexity (random inputs)

Can we solve NP-hard problems on very large instances?

Theoretical Complexity only Bounds the Practical Complexity

Worst-case complexity $\neq$ **average** time-complexity (random inputs)

Can we solve NP-hard problems on very large instances? **Yes!** But not on all (even small) instances (probably exponential)

Practical instances may happen to be easy
— *Polynomial class*
— *Phase transition*
— *Pathological examples* (exponential algorithm with polynomial empirical complexity)

Constraint Satisfaction Problems

Input:

Finite set of **Variables**	$x_1, \dots, x_n$
Corresponding **Domains** of values	$D_1, \dots, D_n$
Finite set of **Constraints**	$c_1, \dots, c_k$
Optional objective function	$f(x_1, \dots, x_n) \in \mathbb{R}$

Output:

Decision	$\exists?(x_1, \dots, x_n) \in D_1 \times \cdots \times D_n$ s.t. $c_1 \wedge \cdots \wedge c_k$
Optimization	$\min_{c_1 \wedge \cdots \wedge c_k} f(x_1, \dots, x_n)$
Solutions	$\operatorname{argmin}_{c_1 \wedge \cdots \wedge c_k} f(x_1, \dots, x_n)$

The N Queens Problem

N Queens on an $N \times N$ chessboard with no attack

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Variables:

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Domains:

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$\forall\, i < j\; x_i \neq x_j$

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Variables: $x_1, \dots, x_N$ (columns)
Domains: $D_i = \{1, \dots, N\}$ (lines)
Constraints: "not same line"
$\forall i < j \; x_i \neq x_j$
"not same diagonal"
$\forall i < j \; x_i \neq x_j + i - j \wedge x_i \neq x_j - i + j$

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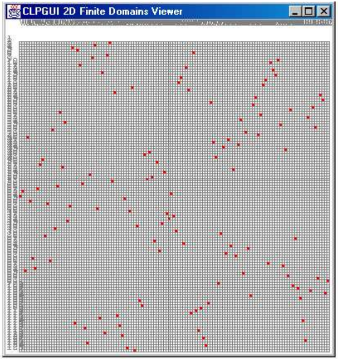


MiniZinc Constraint Modelling Language

```
int: n;
array[1..n] of var 1..n: queens;

constraint forall (i, j in 1..n where i < j) (
   queens[i] != queens[j] /\
   queens[i] != queens[j] + j - i /\
   queens[i] != queens[j] + i - j
);

solve satisfy;
```

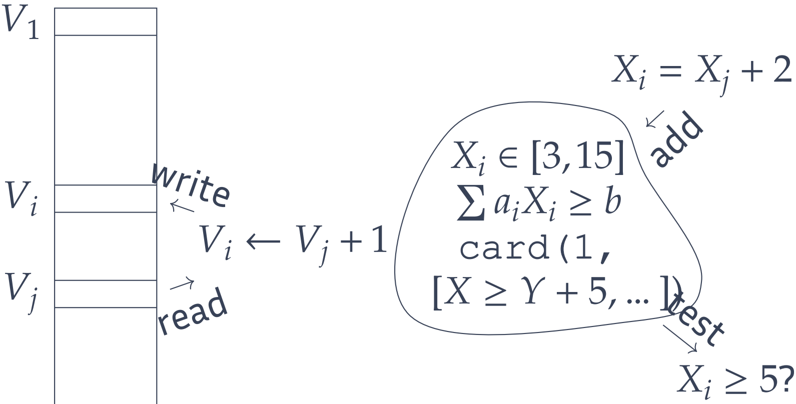
(Declarative) Programming as Modeling

1940 Machine language
1954 Fortran: arithmetic expressions and control flow
1959 Lisp: functions over lists (Church's λ-calculus)
1960 Algol: algorithms
1970 C: for whole operating system
1972 **Prolog**: first-order logic
1975 Smalltalk: objects
1978 ML: typed functions
1984 **Constraint Logic Programming**
1990 Constraint programming libraries for C++
1991 Python
1996 Java: object-oriented threaded programming
2008 **Zinc**: solver-independent constraint language
...

Von Neumann vs. Constraint machine

memory of values
programming variables

memory of constraints
mathematical variables



and-concurrency
or-parallelism

Constraints are Domain Filtering Agents

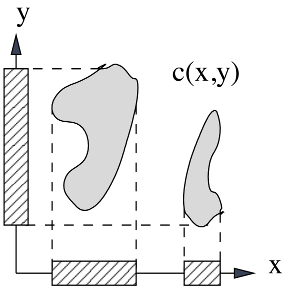
for each constraint c_i

for each variable x_j in c_i

compute the projection P_{ij} of the solutions on x_j

over-approximate if necessary

$D_j \leftarrow D_j \cap P_{ij}$



communication through shared variables in $\wedge$

Or-branches: no communication, easy to parallelize
And-concurrency: lots of comm., difficult to parallelize

Part II

MiniZinc, Jupyter and friends...

MiniZinc

Most of the course will use MiniZinc as programming/modelling language

MiniZinc $\xrightarrow{\text{compile}}$ FlatZinc

high-level low-level

MiniZinc syntax

A MiniZinc program may contain **parameters**

```
int: i = 1;
int j;
j = 2;
```

Fixed value (named constants), of type `int`, `float`, `bool` or `string`

May be given in a separate *data* file (`.dzn`)

MiniZinc syntax

A MiniZinc program may contain **decision variables**

```
var int: u;
var 1..10: v;
```

Only `int` or `float`, given with an optional *domain*

Parameters and variables can appear in **constraints** (using the usual arithmetic and Boolean relation operators)

```
constraint u = 18 * v + 42;
constraint alldifferent([x, y, z]);
```

Additional MiniZinc Syntax

It is possible to define **sets** and **arrays** of objects

```
set of int: STUDENT = 0..n;
% m[i] is the mark of student number i
array[STUDENT] of var int: m;
```

Iterators over those structures are given

```
constraint exists(s in STUDENT)
    (m[s] = 20);
constraint forall(s in STUDENT)
    (m[s] <= 20);
```

Final Bits

At least one **solve** statement must appear in a MiniZinc model

```
solve satisfy;
solve maximize u+3*v;
solve minimize sum(i in STUDENT)(m[i]);
```

output takes a list of strings and displays them

Many other things (function or predicate definition, enums, comprehensions, etc.) $\longrightarrow$ in TDs

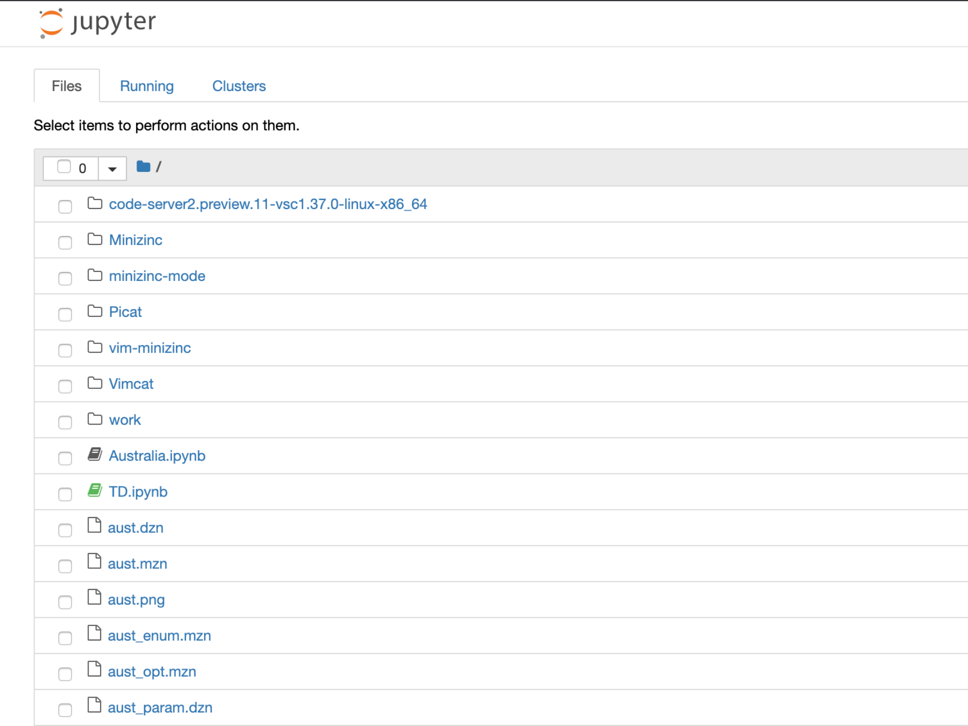
Jupyter

Work will be done using **Jupyter Notebooks**

Assuming some basic knowledge of Python

Otherwise see: `https://docs.python.org/3/tutorial/introduction.html`

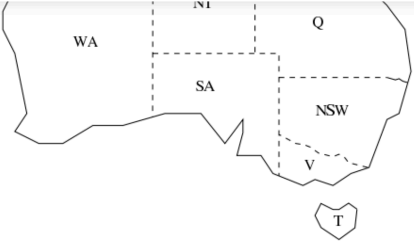
Code editing will use a version of **VS Code** embedded in a browser. (`vim` and `emacs` are also provided)


jupyter
Files
Running
Clusters
Select items to perform actions on them.
0
/
code-server2.preview.11-vsc1.37.0-linux-x86_64
Minizinc
minizinc-mode
Picat
vim-minizinc
Vimcat
work
Australia.ipynb
TD.ipynb
aust.dzn
aust.mzn
aust.png
aust_enum.mzn
aust_opt.mzn
aust_param.dzn

jupyter TD (unsaved changes)

File Edit View Insert Cell Kernel Help Trusted

Run Markdown



A MiniZinc model is composed of

- variable declarations
- constraints
- a `solve` statement

Let us first have a look at a partial model (file `aust.mzn`) for coloring Australia such that two neighbor states have a different color.

As you see, the basic editor is not that great… You can instead open a terminal (from the launcher) and launch vi/emacs if you like.

We will have colorized output in the notebook through a shell command

In [1]:
```
!vimcat.sh aust.mzn
```

```
% Colouring Australia using nc colours
int: nc;

var 1..nc: wa;   var 1..nc: nt;  var 1..nc: sa;   var 1..nc: q;
var 1..nc: nsw;  var 1..nc: v;   var 1..nc: t;

constraint wa != nt;
```

File Edit Selection View Go Debug Terminal Help

EXPLORER

OPEN EDITORS

- inf555.py

JOVYAN [CODE-SERVER]

- .cache
- .conda
- .config
- .empty
- .ipynb_checkpoints
- .ipython
- .jupyter
- .local
- .pylint.d
- .vim
- code-server2.preview.11-vsc1.37.0-linux-x...
- Minizinc
- minizinc-mode
- Picat
- vim-minizinc
- Vimcat
- work
- .bash_logout
- .bashrc
- .emacs
- .profile
- .viminfo
- .wget-hsts
- aust_enum.mzn
- aust_opt.mzn

inf555.py

inf555.py > ...

```python
"""Generic module for using pymzn nicely in a Jupyter notebook.

Used in the class INF555
Written by Sylvain.Soliman@inria.fr
"""
import matplotlib.pyplot as plt

import numpy as np

from pymzn import (
    Solver,
    cbc,
    chuffed,
    rebase_array,
)
from pymzn import minizinc as minizn


def minizinc(mzn, *dzn_files, include=".", **kwargs):
    """Solve using minizinc."""
    return minizn(mzn, *dzn_files, include=include, **kwargs)


class PicatSatSolver(Solver):
    """Solver instance for PicatSat."""

    def __init__(self):
        """Nothing special here."""
        super().__init__(solver_id='picat')

    def args(self, fzn_file, **kwargs):
        """Arguments for the CLI call."""
        return ["picat", "fzn_picat_sat", fzn_file]
```

Docker

All will be run from **Docker** containers

Same environment for everyone

Do not forget to save your work!
upload it to the Moodle at the end of the TD session
(and later...)

Setup

1. Download Docker
   `https://docs.docker.com/install/`

2. Install Docker

3. Pull the image for the course

```
docker pull \
registry.gitlab.inria.fr/soliman/inf555
```

Windows Users

You might need Docker Toolbox (and not CE) if you are using *Family Edition*
`https://docs.docker.com/toolbox/toolbox_install_windows/`

You might need to use `192.168.99.100` instead of `localhost` for connecting to Jupyter

If all else fails, use VirtualBox to install an Ubuntu image and follow the instructions to install Docker there

TD1

Pull the missing files

```
docker pull \
registry.gitlab.inria.fr/soliman/inf555/td1
```

Run on **local port 8888** with the `work` directory of the container pointing to where you launch the command

```
docker run -p 8888:8888  -p 8080:8080 -v \
"$PWD":/home/jovyan/work \
registry.gitlab.inria.fr/soliman/inf555/td1
```

You can now start the TD: `http://localhost:8888/notebooks/TD.ipynb`