Symmetries and Symmetry Breaking

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4. Constraint Logic Programs (CLP) : operational and fixpoint semantics
5. Logical Semantics of CLP, automated deduction
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Symmetries and Symmetry Breaking

1. Examples of symmetries in the N-queens problem
2. General variable symmetry breaking
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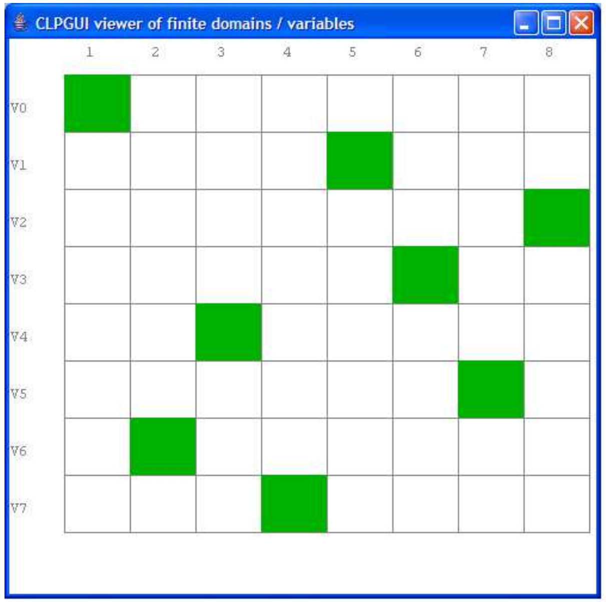
4. Variable and value symmetry breaking
5. Variable-value symmetry breaking

Symmetries in the N-queens problem



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iff

queens(N,[XN,...,X1]) *horizontal axis symmetry*

variable symmetry

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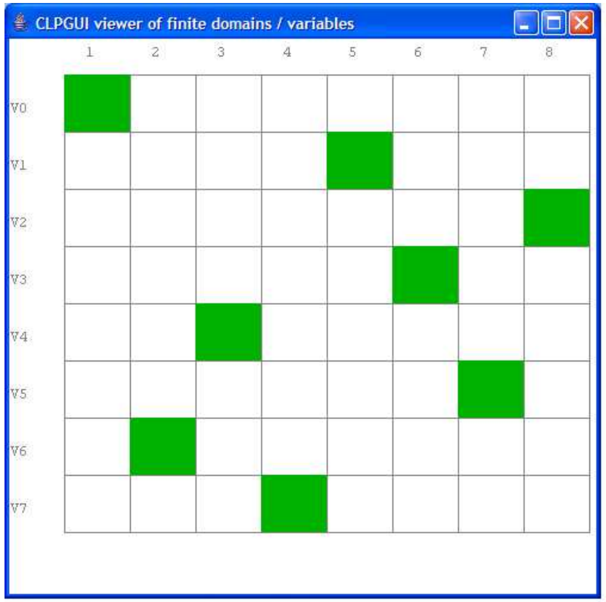

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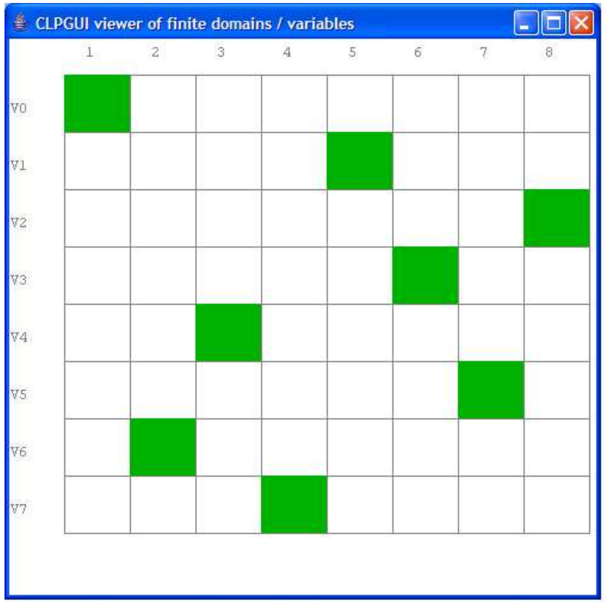

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iff queens(N,[Y1,...,YN]) where Xi=j iff Yj=N+1-i $+90^o$ *rotation symmetry*

variable-value symmetry

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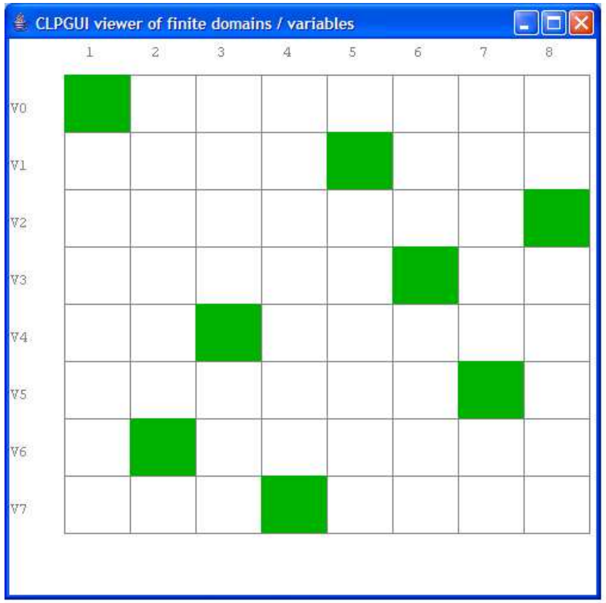

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value symmetry broken by X1¡5

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Variable Symmetries

Given a Constraint Satisfaction Problem (CSP) $c(x_1, ..., x_n)$ over $\mathcal{X}$

a variable symmetry σ is a bijection on variables that preserves solutions:

$$\mathcal{X} \models c(x_1, ..., x_n) \; iff \; \mathcal{X} \models c(x_{\sigma(1)}, ..., x_{\sigma(n)})$$

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Proposition 1 (Crawford et al 96) *If $(\mathcal{X}, \leq)$ is an order, all variable symmetries can be broken by the global constraint*

$$\bigwedge_{\sigma \in \Sigma} [x_1, ..., x_n] \leq_{lex} [x_{\sigma(1)}, ..., x_{\sigma(n)}]$$

Proof: This is one way to choose a unique member in each equivalence class of symmetric assignments. $\square$

Variable Symmetry Breaking

Global constraint $[x_1, ..., x_n] \leq_{lex} [x_{\sigma(1)}, ..., x_{\sigma(n)}]$

arc consistent (AC) if for every variable, every value in its domain belongs to a solution

```
lex(L):-
    lex(L,B),
    B=1.
lex([],1).
lex([_],1).
lex([X,Y|L],R):-
    B #<=> (X #< Y),
    C #<=> (X #= Y),
    lex([Y|L],D),
    R #<=> B #\/ (C #/\ D).
```

$O(mn)$ where m is the maximum domain size [Carlsson Beldiceanu 02]

Breaking Several Variable Symmetries

Proposition 2 (Puget 05, Walsh 06)
$AC(\bigwedge_{\sigma\in\Sigma}[x_1, ..., x_n] \leq_{lex} [x_{\sigma(1)}, ..., x_{\sigma(n)}])$ *is strictly stronger than* $\bigwedge_{\sigma\in\Sigma} AC([x_1, ..., x_n] \leq_{lex} [x_{\sigma(1)}, ..., x_{\sigma(n)}])$.

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Proof: Consider two symmetries (1423) and (1243).

Let $x_1, x_2, x_4 \in \{0,1\}$ and $x_3 = 1$.

We have $AC([x_1,x_2,x_3,x_4] \leq_{lex} [x_4,x_3,x_1,x_2])$

cases	$[x_1$	x_2	x_3	$x_4]$	$\leq_{lex}$	$[x_4$	x_3	x_1	$x_2]$
$x_1 = 0$									
$x_1 = 1$									
$x_2 = 0$									
$x_2 = 1$									
$x_4 = 0$									
$x_4 = 1$									

□

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$x_1=1$	1	0				1	1		
$x_2=0$	0	0				1			
$x_2=1$	0	1				1			
$x_4=0$	0	0				0	1		
$x_4=1$	0					1			

$\square$

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cases	$[x_1$	x_2	x_3	$x_4]$	$\leq_{lex}$	$[x_2$	x_4	x_1	$x_3]$
$x_1=0$	0	0				0	1		
$x_1=1$	1	1	1	1		1	1	1	1
$x_2=0$	0	0				0	1		
$x_2=1$	0	1				1			
$x_4=0$	0					1			
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and $AC([x_1,x_2,x_3,x_4] \leq_{lex} [x_2,x_4,x_1,x_3])$.

However, their conjonction is not AC as there is no solution with $x_4 = 0$.

Indeed, suppose that x4 = 0.

Then the first lex constraint implies x1 = x2 = 0.

And the second lex constraint implies x3 = 0, which is not possible. □

Value Symmetry Breaking

A value symmetry is a bijection σ on values that preserves solutions.

$\{x_i = v_i | 1 \leq i \leq n\}$ is a solution iff $\{x_i = \sigma(v_i) | 1 \leq i \leq n\}$ is a solution

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All value symmetries can be broken by posting for each value symmetry σ

$[x_1, ..., x_n] \leq_{lex} [\sigma(x_1), ..., \sigma(x_n)]$ [Petrie Smith 03]

E.g. Let $\sigma(i) = n + 1 - i$.

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If n is even, the constraint is thus equivalent to $x_1 \leq \frac{n}{2}$

If n is odd, it is equivalent to $x_1 \leq \frac{n+1}{2} \wedge x_1 = \frac{n+1}{2} \implies x_2 \leq \frac{n+1}{2} \wedge ...$

Breaking Several Value Symmetries

Proposition 3 *$AC(\bigwedge_{\sigma\in\Sigma}[x_1,...,x_m] \leq_{lex} [\sigma(x_1),...,\sigma(x_n)])$ is strictly stronger than $\bigwedge_{\sigma\in\Sigma} AC([x_1,...,x_m] \leq_{lex} [\sigma(x_1),...,\sigma(x_n)])$.*

PROOF: Consider the two value symmetries defined by $\sigma_1 = (02)$ and $\sigma_2 = (12)$.

Let $x_1 \in \{0,1\}$, $x_2 \in \{0,2\}$.

We have $AC([x_1,x_2] \leq_{lex} [\sigma_1(x_1),\sigma_1(x_2)])$

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cases	$[x_1,$	$x_2]$	$\leq_{lex}$ $[\sigma_1(x_1),$	$\sigma_1(x_2)]$
$x_1 = 0$				
$x_1 = 1$				
$x_2 = 0$				
$x_2 = 2$				

$\square$

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cases	$[x_1,$	$x_2]$	$\leq_{lex}$	$[\sigma_1(x_1),$	$\sigma_1(x_2)]$
$x_1 = 0$	0			2	
$x_1 = 1$	1	0		1	2
$x_2 = 0$	0	0		2	
$x_2 = 2$	0	2		2	0

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cases	$[x_1,$	$x_2]$	$\leq_{lex}$ $[\sigma_2(x_1),$	$\sigma_2(x_2)]$
$x_1 = 0$	0		0	0
$x_1 = 1$	1		2	
$x_2 = 0$	1	0	2	0
$x_2 = 2$	1	2	2	1

□

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Proposition 3 *$AC(\bigwedge_{\sigma\in\Sigma}[x_1,...,x_m] \leq_{lex} [\sigma(x_1),...,\sigma(x_n)])$ is strictly stronger than $\bigwedge_{\sigma\in\Sigma} AC([x_1,...,x_m] \leq_{lex} [\sigma(x_1),...,\sigma(x_n)])$.*

Proof: Consider the two value symmetries defined by $\sigma_1 = (02)$ and $\sigma_2 = (12)$.

Let $x_1 \in \{0,1\}$, $x_2 \in \{0,2\}$. $AC([x_1,x_2] \leq_{lex} [\sigma_1(x_1),\sigma_1(x_2)])$

and $AC([x_1,x_2] \leq_{lex} [\sigma_2(x_1),\sigma_2(x_2)])$

However the conjunction is not AC as there is no solution such that x2 = 2.

Suppose indeed that $x_2 = 2$.

Then the first lex constraint implies x1 = 0,

but [0 2] is not minimal for the second lex constraint. $\square$

Breaking Several Variable and Value Symmetries

Theorem 1 (Puget 05, Walsh 06) *The constraints*

$[x_1, ..., x_n] \leq_{lex} [x_{\sigma(1)}, ..., x_{\sigma(n)}]$ *for each variable symmetry* $\sigma \in \Sigma$

and $[x_1, ..., x_m] \leq_{lex} [\sigma'(x_1), ..., \sigma'(x_n)]$ *for each value symmetry* $\sigma' \in \Sigma'$

leave at least one assignment in each equivalence class of solutions.

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leave at least one assignment in each equivalence class of solutions.

PROOF: For any assignment ν,

one can pick the lex leader ν_1 of ν under Σ

and then the lex leader ν_2 of ν_1 under Σ'

If ν_2 does not satisfy the lex leader constraint under Σ, iterate.

As the lexicographic orders are well-founded, the process terminates, with an assignment that satisfies all lex leader constraints. □

Breaking Several Variable and Value Symmetries

The iterated lex leader may leave several symmetric assignments.

For example, consider the reflection symmetries on both variables and boolean values. The solutions $[0,1,1]$ and $[0,0,1]$ are symmetric

but satisfy the lex constraints

$$[x_1, x_2, x_3] \leq [x_3, x_2, x_1]$$

and

$$[x_1, x_2, x_3] \leq [\neg x_1, \neg x_2, \neg x_3]$$

Indeed $[0,1,1] \leq [1,1,0]$ and $[0,1,1] \leq [1,0,0]$

$[0,0,1] \leq [1,0,0]$ and $[0,0,1] \leq [1,1,0]$

hence both symmetric solutions $[0,1,1]$ and $[0,0,1]$ are lex leaders.

Variable-Value Symmetries

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Definition 2 *A valuation $[x_1, ..., x_n]$ is admissible for σ iff*
$|\{k \mid x_i = j,\ \sigma(i,j) = (k,l)\}| \ = \ n.$

E.g. In the 3-queens, the assignment $[2, 3, 1]$ is admissible for r90 but not $[2, 3, 3]$.

Remark: If $[x_1, ..., x_n]$ is admissible for σ,

let $\sigma[x_1, ..., x_n]$ be its image under σ

$\sigma[x_1, ..., x_n] = [y_1, ..., y_n]$ where $y_k = l$ whenever $x_i = j$ and $\sigma(i, j) = (k, l)$.

Variable-Value Symmetry Breaking

Proposition 4 *All variable-value symmetries can be broken by posting the constraints* $\bigwedge_{\sigma \in \Sigma} admissible(\sigma, [x_1, ..., x_n]) \wedge [x_1, ..., x_n] \leq_{lex} \sigma[x_1, ..., x_n]$

E.g. In the 3-queens, let $x_1 = 2,\ x_2 \in \{1, 3\},\ x_3 \in \{1, 2, 3\}$

r90$[x_1, ..., x_3]$ prunes $X_3 \neq 2$ for admissibility, and $x_2 \neq 3$, $x_3 \neq 1$ for lex.