On Solving Mixed Shapes Packing Problems by Continuous Optimization with the CMA Evolution Strategy

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***Abstract*—Bin packing is a classical combinatorial optimization problem which has a wide range of real-world applications in industry, logistics, transport, parallel computing, circuit design and other domains. While usually presented as discrete problems, we consider here continuous packing problems including curve shapes, and model these problems as continuous optimization problems with a multi-objective function combining non-overlapping with minimum bin size constraints. More specifically, we consider the covariance matrix adaptation evolution strategy (CMA-ES) with a non-overlapping and minimum size objective function in either two or three dimensions. Instead of taking the intersection area as measure of overlap, we propose other measures, monotonic with respect to the intersection area, to better guide the search. In order to compare this approach to previous work on bin packing, we first evaluate CMA-ES on Korf's benchmark of consecutive sizes square packing problems, for which optimal solutions are known, and on a benchmark of circle packing problems. We show that on square packing, CMA-ES computes solutions at typically 14% of the optimal cost, with the time limit given by the best dedicated algorithm for computing optimal solutions, and that on circle packing, the computed solutions are at 2% of the best known solutions. We then consider generalizations of this benchmark to mixed squares and circles, boxes, spheres and cylinders packing problems, and study a real-world problem for loading boxes and cylinders in containers. These hard problems illustrate the interesting trade-off between generality and efficiency in this approach.**

***Keywords*-bin packing; evolutionary computing; continuous optimization; covariance matrix adaptation; square packing; circle packing; mixed shapes packing**

I. Introduction

Bin packing problems are concerned with finding how to place a given number of objects in a limited space without overlapping. The objective is either to use a minimum number of bins for packing a given list of items, or dually, to cut a maximum number of items in a given piece of material. This kind of problems has a wide range of real-world applications in industry, logistics, transport, parallel computing, circuit design and other domains. In many applications, small improvements in the packing can result in considerable benefits.

Packing is however a hard optimization problem. The complexity of the problem, and the possibility or not of using exact methods, depend upon the geometry of the objects and the constraints imposed. Packing objects of different sizes is also much harder than packing objects of same size since the objects are no longer symmetrical. In one dimension, deciding the existence of a packing of a list of objects of different integer lengths, into k bins of some given length, is already an NP-complete problem [1]. Higher dimensional discrete square packing problems, in two or three dimensions for instance, have the same theoretical complexity, and have been studied extensively for their numerous practical applications [2].

In this paper, in addition to packing problems for polygons [3], we consider continuous packing problems with curve shapes. In addition to the problem of packing circles in a rectangular or circular bin[1] [4], [5], we consider packing problems mixing square and curve shapes, such as polygons and circles, and three dimensional packing problems mixing boxes, spheres and cylinders.

On square packing problems, exact methods have been used to find optimal solutions and prove optimality. For instance, constraint-based methods have been used by Simonis and O'Sullivan in [6] to solve Korf's benchmark of discrete consecutive-square packing problems and prove the optimality of solutions, up to 32 squares in [7].

On curve packing problems, inexact methods like for instance genetic algorithms [8] or hybrid simulated annealing with tabu search methods [5], are usually used to compute suboptimal solutions. On circle packing problems, [4] reports the use of exact methods for problems with circles of same size, and global optimization methods from mathematical software for problems with circles of different sizes for which the optimal solutions are not known.

In this paper, we evaluate the covariance matrix adaptation evolution strategy (CMA-ES[2]) [9], one of the most powerful evolutionary algorithm for continuous optimization with arbitrary objective functions, on hard packing problems including square shapes, curve shapes, mixed square and curve shapes, and also continuous rotations.

In order to measure the overlaps between objects, we show that instead of taking the intersection area as overlap measure, other measures, monotonic with respect to the

[1]The web site http://packomania.com contains benchmarks of such circle packing problems.

[2]https://www.lri.fr/~hansen/cmaes_inmatlab.html

intersection area, can better guide the search. We define such monotonic measures for polygons and circles.

We also propose a challenging benchmark of problems which generalizes both Korf's benchmark of consecutive size square packing problems [3], and a similar benchmark of circle packing problems [4], to mixed squares and circles, and three dimensional problems with cubes, spheres, and cylinders. In addition, we consider a real-world application for loading boxes and cylinders in a container[3].

On problems on which the optimal costs are known, we show that CMA-ES computes solutions at typically 14% of the optimal bin size in time comparable to the best dedicated algorithms for finding optimal solutions. On circle packing, we show that CMA-ES computes solutions at 2% of the best known algorithms in the same time limit. On our real-world application for loading a container with boxes and cylinders, CMA-ES computes valuable solutions in typically less than 15 minutes per run for 59 objects.

These results show that solving packing problems by continuous optimization using monotonic overlap measures provides an interesting trade-off between generality and efficiency, and that CMA-ES in particular succeeds in computing quality packings on very hard problems.

II. Packing by Continuous Optimization

Packing problems, either discrete or continuous, can be modelled as continuous optimization problems where the unknowns are the coordinates of the objects, and the objective function to minimize combines a measure f_o of the overlap between the objects, with a measure f_s of the overall space used for the packing.

It is worth remarking that the overlap measure does not need to be the exact area or volume of the intersection, but can be any positive function equal to zero when there is no overlap, and *monotonic* with respect to the area or volume of the intersection.

By taking as objective function $f = \alpha \cdot f_o + f_s$ with a sufficiently high coefficient α, we ensure that the non-overlapping constraints are enforced before the space used is minimized.

A. Monotonic Measures of Overlap for Guiding the Search

Between two squares, rectangles, triangles or more generally polygons, the overlap can be measured as the area o of the intersection. CGAL[4] is a computational geometry library which can be used for these computations.

However, in the case where one polygon is totally included in another one, the intersection area which is the area of the included polygon, remains equal as long as the polygon is included. In order to guide the search towards the elimination of this case, we found it important to add a measure of the degree of inclusion. We thus take $f_o = o+d$, where d is the sum of the distances between the borders of the pairs of polygons that contain each other. Indeed, d decreases when the included polygons approach the borders of their enclosing polygons, and is equal to zero when there is no more total inclusions.

The overlap between two circles, given by the coordinates of their center and their radius, (x_1, y_1, r_1) and (x_2, y_2, r_2), does not need to be the area of intersection of the circles. We found it preferable to measure the overlap as the positive difference between the sum of their radii and the distance between their centers, i.e. $\max(0, r_1 + r_2 - \sqrt{(x_1 - x_2)^2 + (u_1 - y_2)^2})$. Indeed, in the case of total inclusion of one circle in another, this simpler measure has the advantage of decreasing when the distance between the borders decreases, and thus measures the degree of total inclusion as well.

Similarly, between a square $ABCD$ and a circle $\mathcal{C} = (O, r)$, we use the overlap measure $d_{\text{ps}}(O, ABCD) + d_{\text{sc}}([AB], \mathcal{C}) + d_{\text{sc}}([BC], \mathcal{C}) + d_{\text{sc}}([CD], \mathcal{C}) + d_{\text{sc}}([AD], \mathcal{C})$ where $d_{\text{ps}}(O, ABCD)$ is the distance between point O and the closest edge of square $ABCD$ if O is in $ABCD$, 0 otherwise; and, writing P for be the perpendicular projection of O on (AB), $d_{\text{sc}}([AB], (O, r))$ is the minimum distance $\min(\|PA\|, \|PB\|)$ if P is in $[AB]$, otherwise the positive difference $\max(0, r - \min(\|OA\|, \|OB\|))$. The idea here is again to provide a measure of the degree of total inclusion. This measure is generalized to polygons and circles.

These measures of overlap in two dimensions (2D) can also be generalized to measure the overlap of objects in 3D, between polygons (boxes, prisms,...), spheres, cylinders, and rigid assemblies of 3D objects.

Interestingly, they can also be generalized to allow continuous rotations of objects, by adding an orientation for each object. This can be achieved by adding one angle variable in 2D, or three angle variables in 3D, to each object. Figure 1 depicts some solutions found with these measures using CMA-ES, for packing triangles with continuous rotations in a rectangular bin of minimum area[5].

It is worth noting that in this approach, the capability to define and compute a measure of overlap between objects is the only requirement for packing arbitrary complex shapes, with or without continuous rotations.

B. Measures of the Space Used to Minimize

Now, for minimizing the space used for the packing in the case of a rectangular bin (resp. a box), one can easily take as measure f_s the area (resp. volume) of the enclosing rectangle (resp. box) defined by the extreme points of the objects in the packing.

For a circular bin, as considered in the circle packing benchmark of the next section, one can enforce it to be centered in the origin without loss of generality, and then

[3]All the packing problems presented in this article, and the source code in C are available at http://contraintes.inria.fr/benchmarks/packingCMAES/

[4]http://www.cgal.org/

[5]Friedman's results on packing unit squares with continuous rotation are available at http://www2.stetson.edu/~efriedma/papers/squares/squares.html

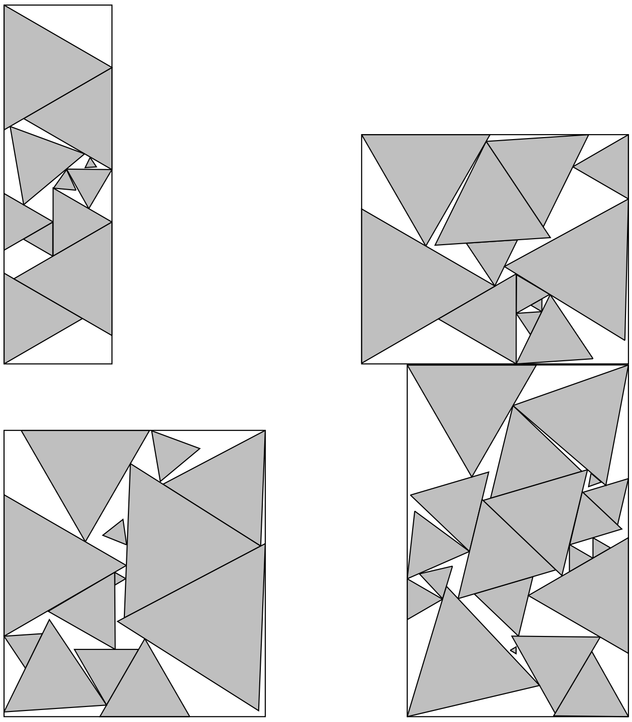

Figure 1. Solutions found by CMA-ES (with four restarts) for packing with rotations N (respectively 11, 12, 13 and 20) equilateral triangles of consecutive sizes from 1 to N, in a rectangular bin of minimum area (here 9.53×31.62, 17.82×20.78, 20.34×22.25 and 32.54×51.65). It is remarkable that the packings in staggered rows are found by continuous optimization.

take as measure f_s of the space used just its radius, defined as the maximum distance of the extreme points of the objects, i.e. for polygons the largest distance to their extreme points, and for circles and spheres, the distance to their center plus radius.

III. Covariance Matrix Adaptation Evolution Strategy

The Covariance Matrix Adaptation Evolution Strategy (CMA-ES[6]) [9] is one of the most powerful global optimization strategy for minimizing an objective function over the reals in a "black-box" scenario, i.e. without assuming any property about the objective function. This method is a multi-point method which uses of population of configurations (here packings defined by the coordinates and orientations of the objects) to sample the search space, estimates the covariance matrix at each sampling, determines the next move in the most promising direction (here translations and rotations of objects), and updates accordingly the multi-variate normal distribution for the next sampling (i.e. mean value and variance of the coordinate and orientation variables).

CMA-ES behaves in effect like a gradient-based method where the gradient is estimated by sampling, according to some multi-variate normal distribution of the variables, which is itself updated during search to adapt to the landscape. When the objective function does not improve, CMA-ES can be restarted to find different local optima. We refer to [9] for more details on that evolutionary algorithm.

The parameters of CMA-ES were used with their default values as follows: a population size of 100 and a stopping criterion based on either an improvement of less than $1e-12$ in the objective function, or a standard deviation of the distribution less than 0.5 for rectangular bin packing (0.01 for the circle packing benchmark), or a fixed timeout indicated in the next section for each benchmark. For packing objects in a rectangle, the initial distribution of solutions was centered on the coordinate value 100 with a standard deviation of 100, so that the initial solutions have few overlaps. For packing circles in a circle, we start from an initial distribution of solutions centered on the origin and a standard deviation of 1, which is sufficient to ensure that the initial placements have few overlaps when the radii are fractional.

IV. Results on Consecutive Square, Circle, Cube, Sphere, and Mixed Packing Problems

In this section, we first consider Korf's benchmark of *Consecutive-Square Packing Problems* [3]. This benchmark consists in finding for each number of squares N, a rectangle of minimum area in which can be packed all the squares of sizes $1 \times 1, 2 \times 2, \cdots, N \times N$ without overlap. In [7], the problem has been solved and optimality proved for all values of $N \leq 32$ using a constraint-based method and a search strategy similar to [6].

Figure 2 first shows an example of suboptimal packing obtained with CMA-ES for 23 squares. Table I summarizes the results obtained on all instances of that problem for $N \leq 32$, and compares them to the best dedicated algorithm of Huang and Korf in [3], [6] for finding the optimal solution and proving optimality. This table shows that with a time credit equal to the computation time of the dedicated algorithm, CMA-ES finds solutions at 14% of the optimal solution in average (the best known exact algorithms find the optimal solution within 240s up to 25 squares).

The second example in Figure 2 is a packing of 16 equilateral triangles of consecutive sizes from 1 to 16 in a rectangle of minimum area. Table II summarizes the results found by CMA-ES for $N \leq 30$.

We also consider the problem of packing N circles of radii $i^{-1/2}$ for $i = 1, ..., N$, in a circle of minimum radius, introduced by Castillo et al. [4]. The third example in Figure 2 is a packing found by CMA-ES for 18 circles. Table III compares the results obtained with CMA-ES within the time used by global optimization software packages dedicated to this problem, namely LINGO [10], NMinimize [11], and MathOptimizer Professional [12]. These results show that CMA-ES computes solutions at 2% from the best known solutions obtained with these algorithms.

One striking feature of dealing with packing problems by continuous optimization, is its ability to handle arbitrary complex shapes, provided one can define a measure

[6] https://www.lri.fr/~hansen/cmaes_inmatlab.html

Table III

RESULTS OBTAINED WITH CMA-ES, LINGO, NMINIMIZE AND MATHOPTIMIZER PRO FOR PACKING N CIRCLES OF RADII $i^{-1/2}$ FOR $1 \leq i \leq N$ IN A BOUNDING CIRCLE OF MINIMAL RADIUS. THE TIME LIMIT USED FOR CMA-ES IS THE MAXIMUM TIME (GIVEN IN SECONDS) USED BY THE OTHER ALGORITHMS. THE BEST SOLUTION FOUND BY CMA-ES IS GIVEN WITH RATIO WITH RESPECT TO BEST KNOWN SOLUTION, AVERAGE COST, STANDARD DEVIATION AND NUMBER OF RESTARTS.

N	CMA-ES				LINGO		NMinimize		MathOptimizer	
	Best radius (Ratio)	Restarts	Average $\pm$ Std dev.	Time	Radius	Time	Radius	Time	Radius	Time
5	1.7518 (1.000)	32	1.7960 ± 0.0288	3	1.7516	3	1.7734	2	1.7516	3
6	1.8117 (1.001)	31	1.8527 ± 0.0246	5	1.8236	5	1.8473	2	1.8101	1
7	1.8501 (1.006)	32	1.9015 ± 0.0244	8	1.8476	8	1.8921	3	1.8387	2
8	1.8638 (0.992)	31	1.9436 ± 0.0337	11	1.9095	11	1.9377	4	1.8796	3
9	1.9367 (1.009)	29	1.9775 ± 0.0300	15	1.9201	15	1.9758	9	1.9221	4
10	1.9590 (1.011)	24	1.9992 ± 0.0248	16	1.9553	16	1.9647	14	1.9382	6
12	2.0110 (1.010)	29	2.0596 ± 0.0368	32	2.0038	31	2.0617	32	1.9902	11
14	2.0486 (1.008)	42	2.0927 ± 0.0302	65	2.0371	43	2.0663	65	2.0316	18
16	2.0658 (1.005)	72	2.1329 ± 0.0302	160	2.0562	78	2.1166	160	2.0661	28
18	2.0983 (1.014)	99	2.1707 ± 0.0323	302	2.1142	90	2.1213	302	2.0700	50
20	2.1487 (1.011)	127	2.2056 ± 0.0290	522	2.1382	144	2.1392	522	2.1255	59
25	2.2007 (1.016)	246	2.2926 ± 0.0350	1913	2.1821	248	2.2521	1913	2.1669	792
30	2.2676 (1.026)	385	2.3746 ± 0.0431	5273	2.2112	906	2.2185	5273	2.2149	263
35	2.3407 (1.036)	552	2.4611 ± 0.0535	12311	2.2645	1367	2.3428	12311	2.2597	464

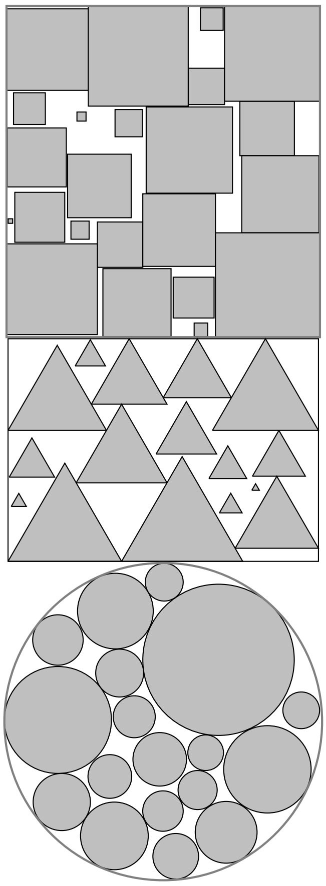

Figure 2. Examples obtained with CMA-ES for packing 23 squares of sizes 1 to 23 in a rectangle of minimum size (here 69×73, see Table I), 16 equilateral triangles of size 1 to 16 in a rectangle (here 41×29.44, see Table II), and for packing 18 circles of radii $i^{-1/2}$ for $1 \leq i \leq 18$ in a circle (here of radius 2.0983, see Table III).

Table I

RECTANGLE OF MINIMUM AREA FOUND BY CMA-ES (GIVEN WITH RATIO W.R.T. THE OPTIMAL SOLUTION [3], AVERAGE COST, STANDARD DEVIATION AND NUMBER OF RESTARTS) FOR PACKING N SQUARES OF SIZES 1 TO N WITHIN A CPU TIME LIMIT OF 240S.

N	Optimal	Best (Ratio)	Average $\pm$ Std dev.	Restarts
10	15×27	15×28 (1.04)	501.07 ± 42.58	346
11	19×27	17×33 (1.09)	655.28 ± 48.37	261
12	23×29	23×30 (1.03)	842.55 ± 67.59	195
13	22×38	27×34 (1.10)	1066.66 ± 82.97	153
14	23×45	32×37 (1.14)	1332.92 ± 99.39	125
15	23×55	27×53 (1.13)	1648.49 ± 121.42	104
16	27×56	39×44 (1.13)	1987.21 ± 153.62	89
17	39×46	35×59 (1.15)	2402.12 ± 169.11	77
18	31×69	35×69 (1.13)	2822.22 ± 202.53	68
19	47×53	39×75 (1.17)	3337.59 ± 297.87	59
20	34×85	55×61 (1.16)	3908.37 ± 315.21	53
21	38×88	50×79 (1.18)	4478.12 ± 360.24	47
22	39×98	58×79 (1.20)	5253.14 ± 456.96	41
23	64×68	69×73 (1.16)	6006.89 ± 468.13	38
24	56×88	69×86 (1.20)	6893.54 ± 481.60	33
25	43×129	78×87 (1.22)	7625.38 ± 577.72	30
26	70×89	73×103 (1.21)	8698.42 ± 610.80	27
27	47×148	71×120 (1.22)	9723.39 ± 905.67	25
28	63×123	94×101 (1.23)	10896.86 ± 876.79	23
29	81×106	77×138 (1.24)	12028.81 ± 694.82	19
30	51×186	70×164 (1.21)	13314.39 ± 1066.88	18
31	91×110	117×118 (1.38)	15239.68 ± 976.47	16
32	85×135	107×134 (1.25)	16480.60 ± 1369.48	15

of overlap, and mixed packing problems combining square and curve shapes. Figure 3 shows suboptimal packings obtained with CMA-ES for packing in a rectangle of minimum area, respectively 16consecutive circles of diameters 1 to 16 with 16 consecutive squares of sizes 1 to 16, 11 consecutive circles with 11 consecutive equilateral triangles, and 14 consecutive squares with 14 equilateral triangles. Tables IV, VI, V summarize our results obtained

Table II
RECTANGLES OF LEAST AREA FOUND BY CMA-ES IN 240S FOR PACKING N TRIANGLES WITH EDGES OF LENGTH 1 TO N.

N	Minimum area	Average $\pm$ Std dev.	Restarts
10	20.50×14.72	341.53 ± 19.83	257
11	24.00×16.45	450.62 ± 27.31	201
12	28.00×18.19	573.96 ± 37.74	159
13	31.50×20.78	730.47 ± 41.79	139
14	27.50×29.44	896.10 ± 48.04	118
15	30.00×32.91	1093.96 ± 58.54	100
16	41.00×29.44	1326.80 ± 69.05	87
17	52.50×26.85	1581.46 ± 90.81	77
18	44.00×38.11	1889.77 ± 121.13	67
19	30.54×64.02	2186.84 ± 103.00	59
20	55.50×40.70	2529.21 ± 114.33	49
21	52.70×50.23	2922.20 ± 144.89	40
22	76.00×40.70	3345.22 ± 132.38	34
23	63.17×55.28	3816.09 ± 235.41	33
24	118.83×34.64	4377.98 ± 232.66	30
25	68.50×64.95	4861.69 ± 256.24	25
26	75.48×66.70	5485.06 ± 376.62	21
27	104.92×53.84	6077.71 ± 254.52	22
28	106.00×61.49	6878.06 ± 329.60	18
29	81.25×86.60	7511.72 ± 366.70	13
30	63.94×120.48	8282.33 ± 370.16	14

Table IV
RECTANGLES FOUND BY CMA-ES IN 240S FOR PACKING N SQUARES OF SIZES 1 TO N WITH N CIRCLES OF DIAMETERS 1 TO N.

$2N$	Minimum area	Average $\pm$ Std dev.	Restarts
20	25×34	977.29 ± 78.28	44
22	26×43	1276.62 ± 117.64	34
24	32×43	1652.61 ± 104.59	23
26	37×50	2081.19 ± 167.94	17
28	35×68	2571.16 ± 145.17	13
30	50×57	3094.66 ± 205.60	10
32	56×62	3649.06 ± 90.39	8
34	37×110	4478.51 ± 327.87	10
36	55×92	5618.45 ± 449.19	6
38	58×100	6381.48 ± 362.71	6
40	75×90	7297.70 ± 267.12	5

with CMA-ES with a timeout of 240 seconds, for packing in a rectangle of least area N consecutive objects of one simple shape with N consecutive objects of one other simple shape, for $5 \leq N \leq 20$.

These benchmarks can also be generalized to 3D, with consecutive cube and sphere packing problems. Table VII summarizes our results on cubes. Similarly, Table VIII shows the results for packing in a box N cylinders of consecutive radii and heights 1 to N in 240 seconds.

In all these benchmarks, CMA-ES stops by reaching the minimum threshold value of 10^{-2} for the standard deviation of the multi-variate normal distribution used for the population, and makes restarts within the given time limit. This means that, although starting from a diverse random population of placements, CMA-ES does converge to a particular placement at each run. The convergence is quite

Table V
RECTANGLES FOUND BY CMA-ES IN 240S FOR PACKING N CONSECUTIVE EQUILATERAL TRIANGLES OF SIZES 1 TO N WITH N CIRCLES OF DIAMETERS 1 TO N.

$2N$	Minimum area	Average $\pm$ Std dev.	Restarts
20	34.33×18.99	708.81 ± 38.96	35
22	41.31×20.53	925.37 ± 42.98	27
24	35.55×30.31	1151.48 ± 41.74	16
26	36.59×39.33	1508.72 ± 61.02	10
28	45.40×38.98	1879.02 ± 87.27	6
30	65.22×31.84	2304.74 ± 153.22	9
32	66.40×38.76	2726.06 ± 100.64	6
34	98.15×31.23	3288.53 ± 138.81	7
36	111.17×33.52	3917.68 ± 211.51	5
38	68.19×63.78	4529.97 ± 185.64	3
40	111.46×46.14	5394.88 ± 190.10	3

Table VI
RECTANGLES FOUND BY CMA-ES IN 240S FOR PACKING N CONSECUTIVE EQUILATERAL TRIANGLES WITH N CONSECUTIVE SQUARES.

$2N$	Minimum area	Average $\pm$ Std dev.	Restarts
20	30.00×23.99	832.73 ± 67.10	53
22	28.00×35.06	1100.74 ± 68.27	40
24	32.11×38.24	1405.19 ± 99.15	29
26	38.00×42.78	1802.38 ± 101.08	25
28	60.05×32.82	2245.35 ± 133.97	20
30	49.31×49.99	2693.60 ± 178.35	16
32	48.15×60.80	3294.26 ± 294.43	13
34	90.36×39.53	3912.83 ± 297.77	8
36	66.17×60.00	4456.62 ± 388.32	7
38	82.50×63.71	5547.24 ± 261.57	6
40	91.16×62.53	5942.15 ± 220.46	3

Table VII
BOXES FOUND WITH CMA-ES IN 240 SECONDS FOR PACKING N CUBES OF SIZES 1 TO N.

N	Minimum volume	Average $\pm$ Std dev.	Restarts
10	$13 \times 17 \times 19$	5069.06 ± 453.41	146
11	$11 \times 20 \times 26$	7328.80 ± 742.87	112
12	$17 \times 21 \times 23$	10296.62 ± 932.35	91
13	$19 \times 23 \times 25$	13948.90 ± 1558.83	74
14	$21 \times 25 \times 27$	18614.56 ± 1975.31	63
15	$25 \times 27 \times 29$	24726.58 ± 2168.30	53
16	$26 \times 29 \times 34$	31467.78 ± 2500.82	44
17	$28 \times 33 \times 35$	40458.57 ± 4431.68	40
18	$30 \times 34 \times 41$	50018.66 ± 4627.22	32
19	$33 \times 36 \times 45$	62959.90 ± 5591.34	28
20	$35 \times 39 \times 50$	77675.50 ± 6638.45	24
21	$36 \times 37 \times 62$	94891.15 ± 8857.45	21
22	$41 \times 42 \times 54$	113875.23 ± 9468.84	17
23	$40 \times 48 \times 62$	132324.31 ± 8588.13	17
24	$40 \times 48 \times 78$	160632.34 ± 9899.59	13
25	$43 \times 49 \times 75$	183628.71 ± 14416.27	13
26	$43 \times 64 \times 72$	226350.20 ± 19656.29	11
27	$54 \times 55 \times 71$	253237.57 ± 23096.49	11
28	$53 \times 67 \times 74$	307933.26 ± 27064.55	9
29	$52 \times 68 \times 90$	336172.32 ± 17303.35	8
30	$57 \times 67 \times 94$	387778.48 ± 34668.24	7

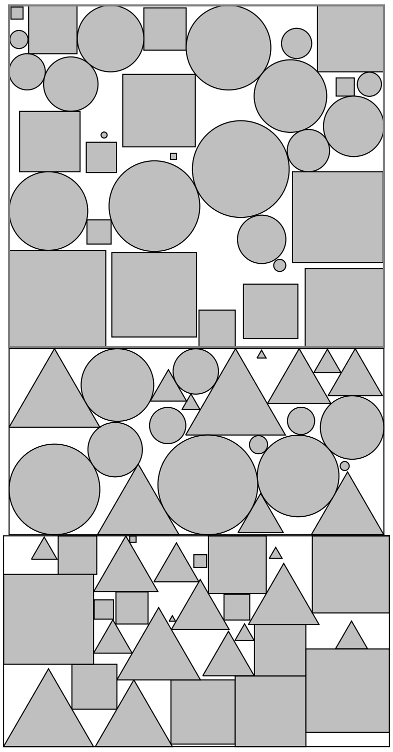

Figure 3. Example of placement obtained with CMA-ES in 240s for packing 16 circles of diameters 1 to 16 with 16 squares of sizes 1 to 16 in a rectangle (here 62×56, see Table IV), 11 circles of diameters 1 to 11 with 11 equilateral triangles of sizes 1 to 11 in a rectangle (here 41.31×20.53, see Table V), and for packing 14 squares of sizes 1 to 14 with 14 equilateral triangles of sizes 1 to 14 in a rectangle (here 60.05×32.82, see Table VI).

Table VIII
BOXES FOUND BY CMA-ES IN 240 SECONDS FOR PACKING N PARALLEL CYLINDERS OF CONSECUTIVE RADII AND HEIGHTS FROM 1 TO N.

N	Minimum volume	Average $\pm$ Std dev.	Restarts
10	$10 \times 37 \times 39$	18181.40 ± 1594.20	266
11	$19 \times 22 \times 49$	25994.71 ± 2091.52	214
12	$12 \times 45 \times 55$	36650.77 ± 2917.79	176
13	$13 \times 41 \times 76$	49515.17 ± 4282.56	146
14	$27 \times 41 \times 47$	67040.26 ± 6270.90	114
15	$27 \times 47 \times 55$	87102.24 ± 7948.93	101
16	$26 \times 57 \times 61$	112985.10 ± 10750.13	83
17	$30 \times 62 \times 65$	141018.32 ± 11130.01	72
18	$30 \times 65 \times 76$	178727.75 ± 14711.77	60
19	$35 \times 59 \times 87$	218760.42 ± 20027.57	51
20	$37 \times 65 \times 96$	269114.06 ± 20665.73	44
21	$36 \times 77 \times 99$	326470.14 ± 27625.47	38
22	$38 \times 81 \times 110$	404863.90 ± 41335.54	33
23	$44 \times 80 \times 118$	468015.13 ± 35350.81	29
24	$63 \times 81 \times 91$	555001.05 ± 37500.15	24
25	$47 \times 98 \times 125$	665798.73 ± 58224.89	21
26	$49 \times 111 \times 120$	753064.33 ± 81645.53	18
27	$65 \times 92 \times 135$	898699.65 ± 66555.36	18
28	$70 \times 110 \times 122$	1053994.27 ± 85033.27	15
29	$79 \times 103 \times 112$	1113128.42 ± 96514.84	12
30	$103 \times 106 \times 110$	1380865.71 ± 121570.06	14

slow but we have checked that the CMA-ES strategy leads to better compactions than algorithmically, by iterating object moves in the different directions successively until a fixpoint is reached. CMA-ES spends time to infer good packing directions from an initially diverse population of packing solutions, gradually eliminate diverse solutions by decreasing the standard deviation of the distribution, and converge to a particular placement for the whole population. However, restarts are necessary to explore other local minima, and one defect of the CMA-ES strategy is that each restart is performed with a complete loss of information from the previous runs.

V. Results on a Real-World Container Loading Problem with Boxes and Cylinders

In logistics, the main problems addressed are packing, design of optimal plan of packing products in cartons and cartons in pallets, optimization in distribution by minimizing the number of pallets, optimization of vehicle or container loading plans, optimization of assignment of containers in wagons. In this section, we present a real-world problem from our industrial company for packing boxes and cylinders in a container.

In a simplification of the problem, all objects are put in a given orientation and one object cannot be above another one. Consequently, the problem can be solved as a two-dimensional problem. A packing solution provides the position of the base of the objects on the rectangular container floor, without overlapping.

In this industrial application, the width of the container is fixed (equal to 248 cm), while the length is variable and must be minimized. The objective function thus consists in finding the smallest length for the container. This is used in turn to determine the types of trucks to use and their numbers. The actual packing in the trucks is then done manually according to the computed global placement, with some margins added to solve rounding problems. Figure 4 depicts one computed placement. One computation with a stopping criterion based on a standard deviation less than 10^{-2} for the variables takes 743s in average. We used 30 restarts to produce that solution.

Additional constraints, such as partial orderings between objects for satisfying multipe delivery orders, can be easily added in this approach, since the only thing to do is to define a measure of violation of those constraints, and add them to the objective function as done for the non-overlapping constraints.

VI. Conclusion and Discussion

We have shown that solving packing problems by continuous optimization, using monotonic overlap measures, provides an interesting trade-off between generality and efficiency, and that CMA-ES in particular succeeds in computing quality packings on very hard problems.

On Korf's benchmark of discrete consecutive sizes square packing problems, for which the optimal costs are known up to 32 squares, we have shown that the solutions computed by CMA-ES are typically at 14% from the

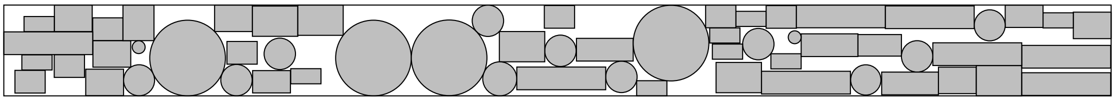

Figure 4. Industrial application: best placement found with 30 restarts for a transportation order of 59 objects mixing boxes and cylinders on a row with a fixed width of 240cm. The computed length of the packing is used to allocate trucks to the order.

optimal cost, with a computation time comparable to the time used by the best exact methods for finding an optimal solution [3].

On a benchmark of consecutive sizes circle packing problems, CMA-ES finds solutions at 2% of the best known costs obtained by running the three global optimization methods reported in Castello et al. [4]. On a generalization of these benchmarks to 3D for cubes and spheres, and to mixed shapes, including square and circles, cubes, spheres and cylinders, as well as on a real-world problem for loading a container with boxes and cylinders, CMA-ES is able to compute similarly suboptimal solutions in less than 240 seconds for problems of size up to 30 objects in 2D. CMA-ES is able to compute a suboptimal solution for the industrial application with 59 objects in less than 800s.

CMA-ES does not perform as well as the best dedicated algorithm restricted to some simple specific shapes, but still performs reasonably well in these particular cases, with the advantage of being able to deal with arbitrary complex shapes in all generality, provided a measure of overlap can be defined. This is remarkable since these packing benchmarks are very challenging problems for black-box continuous optimization methods. The main defect of CMA-ES is probably its sensitivity to the initial sampling and the necessity to restart the method from scratch to escape from local minima. CMA-ES acts as a compactor for a population of random placements, with a subtle effect of "homogeneization" of the population during evolution, which leads to convergence with a small value for standard deviation of the distribution. This evolution strategy leads to better solutions than by applying an algorithmic compactor iteratively in each direction. However it leaves room for improvement concerning the elimination of restarts. In principle, it would indeed be preferable for an evolutionary algorithm to compute several solutions in parallel, rather than sequentially with a complete loss of information at each restart.

We believe that the overlap measures and the consecutive-size packing benchmarks we have presented here are relevant to real-world mixed shapes packing problems, and should be used in the future to evaluate different evolutionary strategies, and measure progress in the field.

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REFERENCES

[1] M. Garey and D. Johnson, *Computers and intractability: a guide to the theory of NP-completeness.* Freeman, New York, 1979.

[2] A. Lodi, S. Martello, and M. Monaci, "Recent advances in two-dimensional bin packing problems," *European Jounral of Operational Research*, vol. 141, no. 2, pp. 241–252, 2002.

[3] E. Huang and R. E. Korf, "Optimal rectangle packing: An absolute placement approach," *Journal of Artificial Intelligence Research*, vol. 46, pp. 47–87, 2012.

[4] I. Castillo, F. J. Kampas, and J. D. Pintér, "Solving circle packing problems by global optimization: Numerical results and industrial applications," *European Journal of Operational Research*, vol. 191, no. 3, pp. 786–802, 2008.

[5] D. Zhang and A. Deng, "An effective hybrid algorithm for the problem of packing circles into a larger containing circle," *Computers & Operations Research*, vol. 32, no. 8, pp. 1941–1951, 2005.

[6] H. Simonis and B. O'Sullivan, "Search strategies for rectangle packing," in *Proceedings of CP'08*, ser. LNCS, P. J. Stuckey, Ed., vol. 5202. Springer-Verlag, 2008, pp. 52–66.

[7] E. Huang and R. E. Korf, "New improvements in optimal rectangle packing," in *Proceedings of the 21st International Joint Conference on Artificial Intelligence, IJCAI'09*, C. Boutilier, Ed., 2009, pp. 51–516.

[8] E. Hopper and B. Turton, "Application of genetic algorithms to packing problems - a review," in *Proceedings of the Second On-line World Conference of Soft Computing in Engineering Design and Manufacturing*, 1997, pp. 279–288.

[9] N. Hansen and A. Ostermeier, "Completely derandomized self-adaptation in evolution strategies," *Evolutionary Computation*, vol. 9, no. 2, pp. 159–195, 2001.

[10] Lindo Systems, "LINGO (release version 9.0)," Lindo Systems, Tech. Rep., 2004.

[11] Wolfram Research, "Mathematica (release version 5.2)," Wolfram Research Inc., Tech. Rep., 2005.

[12] J. Pintér and F. Kampas, "Mathoptimizer professional: key features and illustrative applications," in *Gloval Optimization: from Theory to Implementation*, L. Liberti and N. Maculan, Eds. New York: Springer Science and Business Media, 2006, pp. 263–280.