Games on strings with a limited ordering

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Outline of the presentation

- Basics on EF-Games
- Remoteness
- Labeled $<_p$ structures
- Local games on $<_p$ structures
- Local games on labeled $<_p$ structures
- Global games on labeled $<_p$ structures
- Efficient algorithm to compute remoteness
- Conclusions and future work

EF-Games

- (Logical) combinatorial games
- The playground: two relational structures $\mathcal{A}$ and $\mathcal{B}$ (over the same finite vocabulary)
- Two players: *Spoiler* and *Duplicator*
- Move by Spoiler: select a structure and pick an element in it
- Move by Duplicator: pick an element in the opposite structure
- Round: a move by Spoiler followed by a move by Duplicator
- Game: sequence of rounds
- Duplicator tries to imitate Spoiler

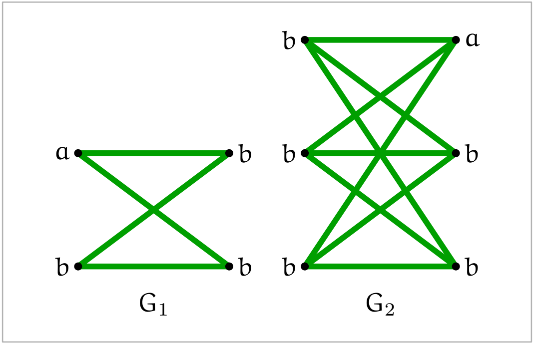
Winning strategies

- Configuration: $((\mathcal{A}, \vec{a}), (\mathcal{B}, \vec{b}))$, with $|\vec{a}| = |\vec{b}|$
  - Represents the relation $\{ (a_i, b_i) \mid 1 \leq i \leq |\vec{a}| \}$
- A play from $((\mathcal{A}, \vec{a}), (\mathcal{B}, \vec{b}))$ proceeds by extending the initial configuration with the pair of elements chosen by the two players, e.g.
  - if Spoiler picks c in $\mathcal{A}$,
  - and Duplicator replies with d in $\mathcal{B}$,
  - then the new configuration is $((\mathcal{A}, \vec{a}, c), (\mathcal{B}, \vec{b}, d))$
- Ending condition: a player repeats a move or the configuration is not a partial isomorphism

Definition

Duplicator has a winning strategy from $((\mathcal{A}, \vec{a}), (\mathcal{B}, \vec{b}))$ if every configuration of the game until an ending configuration is reached is a partial isomorphism, no matter how Spoiler plays.

An example on graphs



- Duplicator must respect the adjacency relation. . .
- . . . and pick nodes with the same label as Spoiler does

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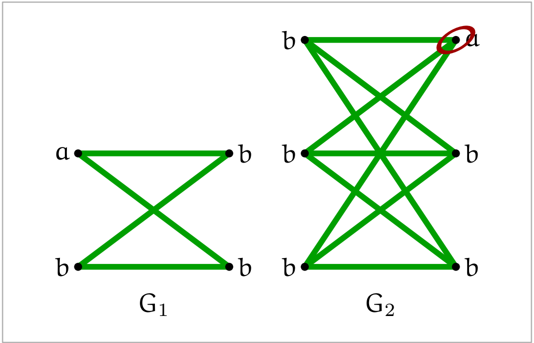

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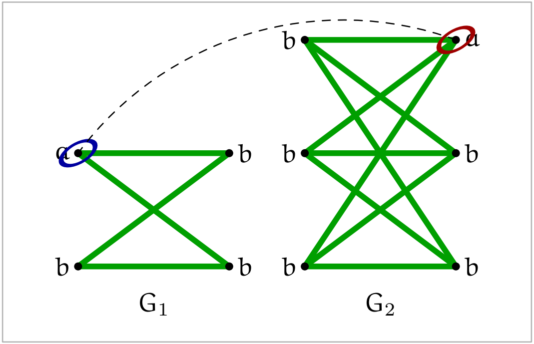

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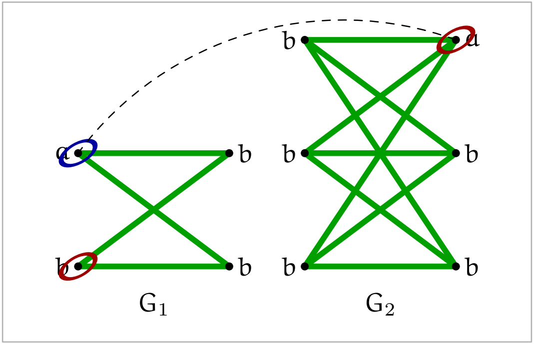

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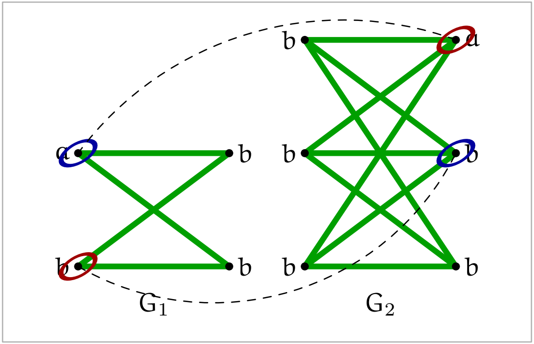

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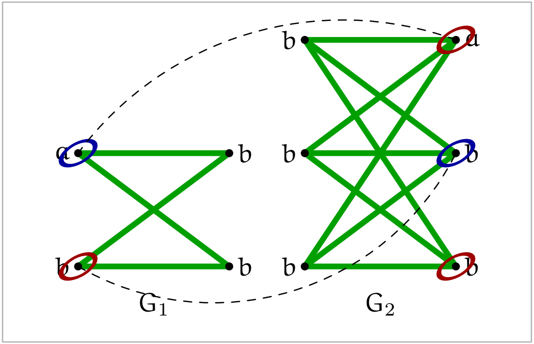

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Bounded and unbounded games

- Bounded game: $\mathcal{G}_m(\mathcal{A}, \mathcal{B})$
- The number m of rounds is fixed
- The game ends after m rounds have been played
- Unbounded game: $\mathcal{G}(\mathcal{A}, \mathcal{B})$
- The game goes on as long as either a player repeats a move, or the current configuration in not partial isomorphism
- Duplicator wins iff the final configuration is a partial isomorphism

Winning and optimal strategies

Winning strategy $\neq$ Optimal strategy

- In unbounded EF-games, Spoiler wins unless $\mathcal{A} \cong \mathcal{B}$
- “Play randomly” is a winning strategy for Spoiler
- But, how far actually is the end of a game?
- What are the *best* (optimal) moves?

Remoteness

- Remoteness of $\mathcal{G}$: the minimum m such that Spoiler wins $\mathcal{G}_m$
  - Simplified definition under the hypothesis $\mathcal{A} \not\cong \mathcal{B}$
- Optimal Spoiler's move: whatever Duplicator replies, the remoteness decreases
- Optimal Duplicator's move: no matter how Spoiler has played, the remoteness decreases at most by 1

Main uses of EF-games

- Prove inexpressibility results (Ehrenfeucht's theorem)
- Establish normal forms for logics (Gaifman's theorem)
- Prove elementary equivalence (Hanf's theorem) and m-equivalence (Sphere lemma) of structures
- Determine how and where two structures differ: use of remoteness to measure the degree of similarity between two structures

Our aim

Compare biological sequences

A simple example

Consider the following two sequences:

agggagtttttaga   *agttagtttagaaggga*

The standard left-to-right comparison:

a	*g*	*g*	*g*	*a*	*g*	*t*	*t*	*t*	*t*	*t*	*a*	–	–	–	–	*g*	*a*
|	|			|	|	|	|	|			|					|	|
a	*g*	*t*	*t*	*a*	*g*	*t*	*t*	*t*	*a*	*g*	*a*	*a*	*g*	*g*	*g*	*g*	*a*

A more flexible way of comparing sequences:

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|	|			|	|	|	|	|			|					|	|
a	*g*	*t*	*t*	*a*	*g*	*t*	*t*	*t*	*a*	*g*	*a*	*a*	*g*	*g*	*g*	*g*	*a*

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|	|			|	|	|	|	|			|					|	|
a	*g*	*t*	*t*	*a*	*g*	*t*	*t*	*t*	*a*	*g*	*a*	*a*	*g*	*g*	*g*	*g*	*a*

A more flexible way of comparing sequences:

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Labeled successor structures

Definition

Let Σ be a fixed alphabet and $w \in \Sigma^*$. A *labeled successor structure* is a pair $(w, \mathbf{i}^n)$ where

- $w = (\{1, \ldots, |w|\}, succ, (P_a)_{a \in \Sigma})$
- $(i, j) \in succ$ iff $j = i + 1$ for all $i, j \in \{1, \ldots, |w|\}$
- $i \in P_a$ iff $w[i] = a$ for all $i \in \{1, \ldots, |w|\}$
- $\mathbf{i}^n$ are distinguished positions $i_1, \ldots, i_n \in \{1, \ldots, |w|\}$

- Necessary and sufficient conditions for Duplicator to win $\mathcal{G}_q((w, \mathbf{i}^n), (w', \mathbf{j}^n))$
- Computation of remoteness in polynomial time using suffix trees (LPAR 2005, GAMES 2007)

The relation $<_p$

What about the linear order relation $<$?

Locality is destroyed :(

We introduce a limited order relation ($<_p$) that lies in between the successor and the linear order relations:

$$i <_p j \text{ iff } i < j \text{ and } j - i \leq p$$

The successor relation and the linear order relation are recovered as special cases of the limited order relation for $p = 1$ and $p = \infty$, respectively

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Our contribution

- Necessary and sufficient conditions for Duplicator to win $\mathcal{G}_q((w, \mathbf{i}^n), (w', \mathbf{j}^n))$ on *labeled* $<_p$ *structures*
- Algorithm to compute the remoteness in polynomial time

Local and global strategy

- Local strategy: how Duplicator must reply when Spoiler plays in the neighborhoods of already selected positions
- Global strategy: how Duplicator must reply when Spoiler plays far from already selected positions

Local games on $<_p$ structures: *pstep*-safety

pstep: "signed distance" between two positions in terms of the number of intervals of length p separating them
Let $i, j, k, p \in \mathbb{N}$, with $i, j, p > 0$ and $k \geq p$.

$$pstep_k^{(p)}(i,j) = \begin{cases} 0 & \text{if } i = j \\ \lceil \frac{j-i}{p} \rceil & \text{if } |i-j| \leq k \text{ and } i < j \\ \lfloor \frac{j-i}{p} \rfloor & \text{if } |i-j| \leq k \text{ and } i > j \\ \infty & \text{if } |i-j| > k \end{cases}$$

A configuration $(w, w', \mathbf{i}^n, \mathbf{j}^n)$ is *pstep-safe* in the k-horizon if $pstep_k^{(p)}(i_r, i_s) = pstep_k^{(p)}(j_r, j_s)$ for all $r, s \in \{1 \ldots n\}$

Lemma

Let $w, w' \in \Sigma^\star$. If $(w, w', \mathbf{i}^n, \mathbf{j}^n)$ is not pstep-safe in the $(p \cdot 2^q)$-horizon, then Spoiler wins $\mathcal{G}_q((w, \mathbf{i}^n), (w', \mathbf{j}^n))$.

Example of *pstep*-safety

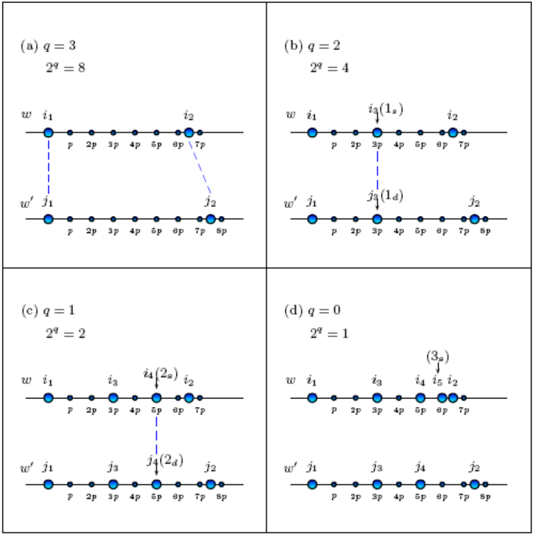


Figure: *pstep*-safety.

Local games on $<_p$ structures: θ-safety

ϑ_k: truncated signed distance between two positions
Let $i, j, k \in \mathbb{N}$, with $i, j, k > 0$.

$$\vartheta_k(i,j) = \begin{cases} i - j & \text{if } |i - j| \leq k \\ \infty & \text{otherwise} \end{cases}$$

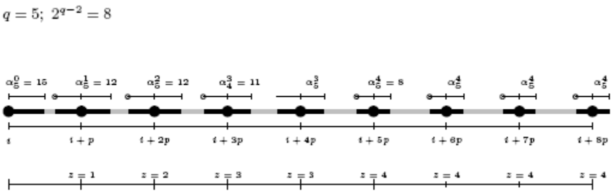
A configuration $(w, w', \mathbf{i}^n, \mathbf{j}^n)$ is *ϑ-safe* in the k-horizon if $\vartheta_k(i_r, i_s) = \vartheta_k(j_r, j_s)$ for all $r, s \in \{1 \ldots n\}$

Lemma

Let $w, w' \in \Sigma^\star$ and $q > 0$. If $(w, w', \mathbf{i}^n, \mathbf{j}^n)$ is not ϑ-safe in the $(2^q - 1)$-horizon, then Spoiler wins $\mathcal{G}_q((w, \mathbf{i}^n), (w', \mathbf{j}^n))$.

Rigid and elastic intervals

- The neighborhood of each position can be partitioned in rigid and elastic intervals (each position origins $2^{q-2}+1$ right and $2^{q-2}+1$ left q-rigid intervals)
- 0th q-rigid interval induced by i: $\rho^{+}_{0,q}(i) = \rho^{-}_{0,q}(i) = [i - \alpha^0_q, i + \alpha^0_q]$, where $\alpha^0_q = 2^{q-1} - 1$
- kth right q-rigid interval induced by i, with $0 < k \leq 2^{q-2}$: $\rho^{+}_{k,q}(i) = (c - \alpha^z_q, c + \alpha^z_q]$, where $c = i + kp$ and α^z_q depends on q and on $z = \lceil \log_2 k \rceil + 1$.



Local games on $<_p$ structures: *p-int*-safety

Definition

Let $q > 0$. A configuration $(w, w', \mathbf{i}^n, \mathbf{j}^n)$ is *p-int-safe* in the k-horizon if for all $r, s \in \{1, \ldots, n\}$, with $r < s$, if there exists $0 \leq h \leq 2^{k-1}$ such that $i_s \in \rho^+_{h,k+1}(i_r)$ or $j_s \in \rho^+_{h,k+1}(j_r)$, then $i_s - i_r = j_s - j_r$

Lemma

Let $w, w' \in \Sigma^\star$ and $q > 0$. If $(w, w', \boldsymbol{i}^n, \boldsymbol{j}^n)$ is not p-int-safe in the q-horizon, then Spoiler wins $\mathcal{G}_q((w, \boldsymbol{i}^n), (w', \boldsymbol{j}^n))$.

Remark

If $(w, w', \mathbf{i}^n, \mathbf{j}^n)$ is *p-int-safe* in the q-horizon, with $q > 0$, then it is ϑ-*safe* in the $(2^q - 1)$-horizon.

Local games on labeled $<_p$ structures

Definition

Let $w \in \Sigma^*$, $q, p \in \mathbb{N}$, with $p > 1$, and $i \in \mathbb{Z}$. The q-color of position i in w, denoted by $q\text{-}col_w(i)$, is inductively defined as follows:

- the 0-color of i in w is the label $w[i]$;
- the (q+1)-color of i in w is the label $w[i]$ plus the q-color of each of the 2^q right intervals and of the 2^q left intervals induced by i.

The *q-color* of the jth right interval $[a, b]$ induced by i, with $1 \leq j \leq 2^q$, is the ordered tuple

$$t_a^w \dots t_{a+\gamma_1-1}^w \{t_{a+\gamma_1}^w \dots t_{b-\gamma_2}^w\} t_{b-\gamma_2+1} \dots t_b^w,$$

where for all $a \leq i \leq b$, t_i^w =q-col$_w(i)$ and γ_1 and γ_2 depend on the radius of rigid intervals.

$<_p$-safety for q-colors

Definition

Let $w, w' \in \Sigma^*$ and $p, n, q \in \mathbb{N}$, with $p > 0$. A configuration $(w, w', \mathbf{i}^n, \mathbf{j}^n)$ is *$<_p$-safe for q-colors* if for all $r \in \{1, \ldots, n\}$, q-col$_w(i_r) =$q-col$_{w'}(j_r)$

Lemma

Let $w, w' \in \Sigma^\star$, and $p, q \in \mathbb{N}$, with $p > 1$. If $(w, w', \boldsymbol{i}^n, \boldsymbol{j}^n)$ is not $<_p$-safe for q-colors, then Spoiler wins $\mathcal{G}_q((w, \boldsymbol{i}^n), (w', \boldsymbol{j}^n))$.

Example of $<_p$-safety for q-colors

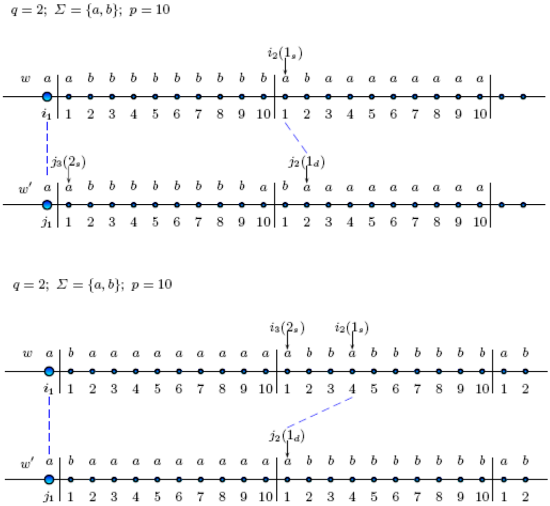


Figure: Safety for q-colors.

Main result (for the local case)

Definition

A configuration $(w, w', \mathbf{i}^n, \mathbf{j}^n)$ is *q-locally-safe* if it is *pstep-safe* in the $(p \cdot 2^q)$-horizon, *p-int-safe* in the *q*-horizon, and $<_p$*-safe for q-colors*.

Theorem

[Sufficient condition for Duplicator to win]
Let $w, w' \in \Sigma^\star$*, and* $p, q \in \mathbb{N}$*, with* $p > 1$*. If* $(w, w', \boldsymbol{i}^n, \boldsymbol{j}^n)$ *is q-locally-safe, then Duplicator wins* $\mathcal{G}_q((w, \boldsymbol{i}^n), (w', \boldsymbol{j}^n))$*.*

Global games on labeled $<_p$ structures

The two strings must have the same q-colors and, for each color, the same multiplicity and a similar distribution.

Let $q, p \in \mathbb{N}^+$, $\mathbf{i}^n$ be a set of positions in w and τ be a $(q-1)$-color.

- $P_{(q,p)}^{(w,\mathbf{i}^n)} = \{j | \ (q-1)\text{-}color_w(j) = \tau \wedge j$ falls "far" from $\mathbf{i}^n\}$
- q-multiplicity: $\rho_{(q,p)}^{(w,\mathbf{i}^n)}(\tau) = |P_{(q,p)}^{(w,\mathbf{i}^n)}|$
- k-scattered set S: $|a - b| > k$ for all $a, b \in S$
- q-scattering $\sigma_{(q,p)}^{(w,\mathbf{i}^n)}(\tau)$: maximal cardinality of a $(p2^q)$-scattered subset of $P_{(q,p)}^{(w,\mathbf{i}^n)}$
- $\Delta_{(w',\mathbf{j}^n)}^{(w,\mathbf{i}^n)} = \{\tau \mid \tau \text{ is a (q-1)-color}, q > 0, \text{ and } \sigma_{(q,p)}^{(w,\mathbf{i}^n)}(\tau) \neq \sigma_{(q,p)}^{(w',\mathbf{j}^n)}(\tau) \vee \rho_{(q,p)}^{(w,\mathbf{i}^n)}(\tau) \neq \rho_{(q,p)}^{(w',\mathbf{j}^n)}(\tau)\}$.

Main result (for the global case)

Theorem

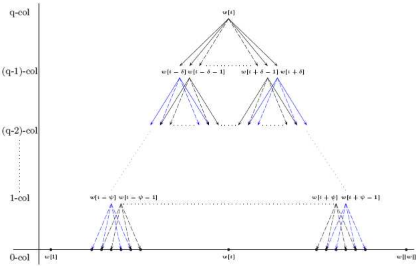
[Main Theorem]
Let $w, w' \in \Sigma^*$ and $p, q \in \mathbb{N}$, with $p > 1$. Duplicator wins $\mathcal{G}_q((w, \boldsymbol{i}^n), (w', \boldsymbol{j}^n))$ if and only if the following conditions hold:

1. $(w, w', \boldsymbol{i}^n, \boldsymbol{j}^n)$ *is q-locally-safe;*
2. *for all $(r-1)$-color $\tau \in \Delta^{(w,\boldsymbol{i}^n)}_{(w',\boldsymbol{j}^n)}$, with $1 \leq r \leq q$, $\sigma^{(w,\boldsymbol{i}^n)}_{(i,p)}(\tau) > q - r$ and $\sigma^{(w',\boldsymbol{j}^n)}_{(i,p)}(\tau) > q - r$.*

Remoteness of $\mathcal{G}$: $r + \min(\sigma^{(w,\mathbf{i}^n)}_{(r,p)}, \sigma^{(w',\mathbf{j}^n)}_{(r,p)})$.

Complexity of remoteness

- Compute in polynomial time scattering and multiplicity of a q-color in a string ($O(p^2 n^3 \log n)$)
- Compare in polynomial time two q-colors ($O(p^2 n^3 \log n)$)
- Each q-color is represented by a layered directed graph
- Bottom-up visit of the graphs



Conclusions and future work

- We analyzed *EF-games* on labeled $<_p$ structures.
- We identified necessary and sufficient winning conditions for Spoiler and Duplicator, that allow one to compute the remoteness of a game and optimal strategies for both players.
- Next step: extensive experimentation of the proposed games on real biological data.


THANKS

Basic definitions

- Vocabulary: finite set of relation symbols
- $\mathcal{A}$ and $\mathcal{B}$ structures on the same vocabulary
- $\vec{a} = a_1, \ldots, a_k \in \mathrm{dom}(\mathcal{A})$
- $\vec{b} = b_1, \ldots, b_k \in \mathrm{dom}(\mathcal{B})$
- Configuration: $((\mathcal{A}, \vec{a}), (\mathcal{B}, \vec{b}))$, with $|\vec{a}| = |\vec{b}|$
  - Represents the relation $\{ (a_i, b_i) \mid 1 \leq i \leq |\vec{a}| \}$

Definition

$((\mathcal{A}, \vec{a}), (\mathcal{B}, \vec{b}))$ is a partial isomorphism if it is an isomorphism of the substructures induced by $\vec{a}$ and $\vec{b}$, respectively.

Main result

First-order EF-games capture m-equivalence

Theorem (Ehrenfeucht, 1961)

Duplicator has a winning strategy in $\mathcal{G}_m((\mathcal{A}, \vec{a}), (\mathcal{B}, \vec{b}))$ if and only if $(\mathcal{A}, \vec{a})$ and $(\mathcal{B}, \vec{b})$ satisfy the same FO-formulas of quantifier rank m and at most $|\vec{a}|$ free variables, written $(\mathcal{A}, \vec{a}) \equiv_m (\mathcal{B}, \vec{b})$.

Corollary

A class $\mathcal{K}$ of structures (on the same finite vocabulary) is FO-definable if and only if there is $m \in \mathbb{N}$ such that Spoiler has a winning strategy whenever $\mathcal{A} \in \mathcal{K}$ and $\mathcal{B} \notin \mathcal{K}$.

Expressiveness results

Exploiting the corollary, we can prove negative expressiveness results.

Example

Let $\mathcal{L}_k \stackrel{\text{def}}{=} (\{1, \ldots, k\}, <)$. It is known that

$$n = p \text{ or } n, p \geq 2^m - 1 \Rightarrow \text{Duplicator wins } \mathcal{G}_m(\mathcal{L}_n, \mathcal{L}_p)$$

"The class of linear orderings of even cardinality is not FO-definable"

- Given m, choose $\tilde{n} = 2^m$ and $\tilde{p} = 2^m + 1$;
- then, Duplicator wins $\mathcal{G}_m(\mathcal{L}_{\tilde{n}}, \mathcal{L}_{\tilde{p}})$ (i.e., $\mathcal{L}_{\tilde{n}} \equiv_m \mathcal{L}_{\tilde{p}}$)

Example of ϑ-safety

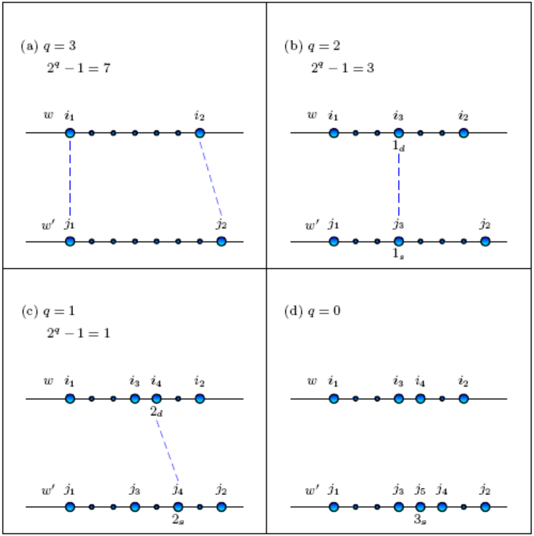
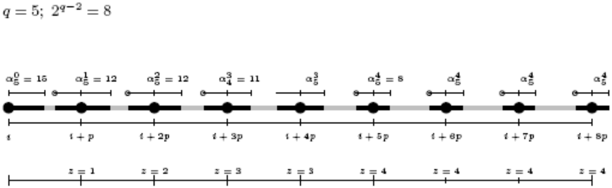


Figure: ϑ-safety.

Rigid and elastic intervals

Definition

Let $q > 1$ and $i \in \mathbb{N}$. The 0th q-rigid interval induced by position i is the closed interval $\rho^{+}_{0,q}(i) = \rho^{-}_{0,q}(i) = [i - \alpha^{0}_{q}, i + \alpha^{0}_{q}]$, where $\alpha^{0}_{q} = 2^{q-1} - 1$. The kth *right (resp., left) q-rigid interval induced by position i*, with $0 < k \leq 2^{q-2}$, is the interval
$\rho^{+}_{k,q}(i) = (c - \alpha^{z}_{q}, c + \alpha^{z}_{q}]$ (resp., $\rho^{-}_{k,q}(i) = [c - \alpha^{z}_{q}, c + \alpha^{z}_{q})$)
where $c = i + kp$ (resp., $c = i - kp$) and
$\alpha^{z}_{q} = 1 + \sum_{j=z-1}^{q-2}(2^{j} - 1)$, where $z = \lceil \log_2 k \rceil + 1$.



Example of *p*-*int*-safety

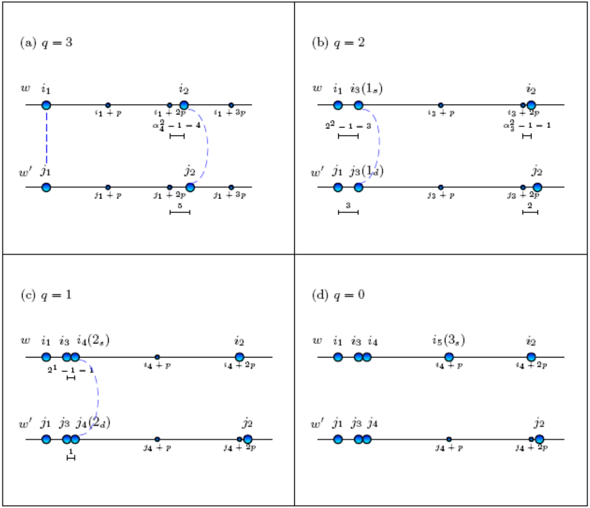


Figure: *p*-*int*-safety.

Local games on labeled $<_p$ structures

Definition

Let $w \in \Sigma^*$, $q, p \in \mathbb{N}$, with $p > 1$, and $i \in \mathbb{Z}$. The q-color of position i in w, denoted by $q\text{-}col_w(i)$, is inductively defined as follows:

- the 0-color of i in w is the label $w[i]$;
- the (q+1)-color of i in w is the ordered tuple $\sigma^w_{2^q} \cdots \sigma^w_1 w[i] \tau^w_1 \ldots \tau^w_{2^q}$ where, for all $1 \leq j \leq 2^q$, τ^w_j (resp., σ^w_j) is the q-color of the j-th right (resp., left) interval induced by i.

The *q-color* of the jth right (resp., left) interval $[a, b]$ induced by i, with $1 \leq j \leq 2^q$, is the ordered tuple $t^w_a \ldots t^w_{a+\gamma_1-1}\{t^w_{a+\gamma_1} \ldots t^w_{b-\gamma_2}\}t_{b-\gamma_2+1} \ldots t^w_b$ (resp., $t^w_a \ldots t^w_{a+\gamma_2-1}\{t^w_{a+\gamma_2} \ldots t^w_{b-\gamma_1}\}t_{b-\gamma_1+1} \ldots t^w_b$)), where for all $a \leq i \leq b$, t^w_i $=$q-col$_w(i)$ and γ_1 and γ_2 depend on the radius of rigid intervals.

Example of safety for q-colors

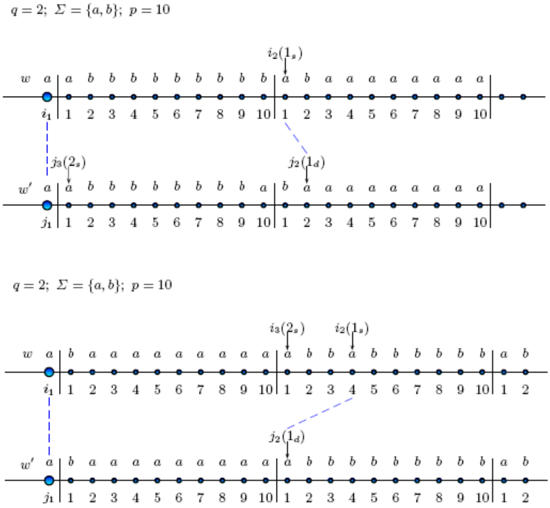
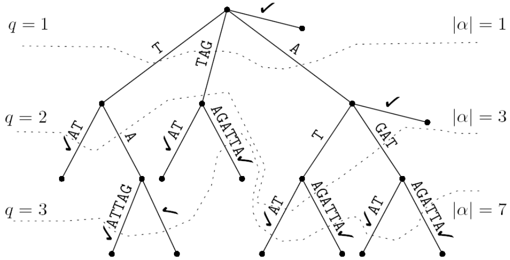


Figure: Safety for q-colors.

Suffix trees



- Let $n = |w| + |w'|$
- Multiplicity values can be computed in $O(n)$ time
- Scattering values can be computed in $O(n \log n)$ time

Global games on labeled $<_p$ structures (1)

The two strings must have the same q-colors and, for each color, the same multiplicity and a similar distribution.

Let $P \subseteq \mathbb{N}$ be a finite set. A *k-blurred partition* $\mathcal{P}$ of P is a partition of P such that (i) for each $A \in \mathcal{P}$ and for each $a, b \in A$, $\delta(a, b) \leq k$, and (ii) there is not a partition $\mathcal{P}'$ satisfying (i) such that $|\mathcal{P}| > |\mathcal{P}'|$. The number of classes of $\mathcal{P}$ is called *k-blurring*.
Let $q, p \in \mathbb{N}^+$, $\mathbf{i}^n$ be a set of positions in w and τ be a $(q-1)$-color.
$\rho^{(w,\mathbf{i}^n)}_{(q,p)}(\tau)$: number of occurrences of τ which are "far" from $\mathbf{i}^n$
$\sigma^{(w,\mathbf{i}^n)}_{(q,p)}(\tau)$: $(p2^q)$-blurring of occurrences of τ which are "far" from $\mathbf{i}^n$
$\Delta^{(w,\mathbf{i}^n)}_{(w',\mathbf{j}^n)} = \{\tau \mid \tau \text{ is a (q-1)-color}, q > 0, \text{ and } \sigma^{(w,\mathbf{i}^n)}_{(q,p)}(\tau) \neq \sigma^{(w',\mathbf{j}^n)}_{(q,p)}(\tau) \vee \rho^{(w,\mathbf{i}^n)}_{(q,p)}(\tau) \neq \rho^{(w',\mathbf{j}^n)}_{(q,p)}(\tau)\}$.